CSE 311 Foundations of Computing I

Lecture 27
Computability: Other Undecidable Problems
Spring 2013

Announcements

- Reading
  - 7th edition: p. 201 and 13.5
  - 6th edition: p. 177 and 12.5
- My office hours this week
  - Usual: today immediately after class until 2:50pm
  - Extra office hour: Thursday 11-12
- Homework 8 due Friday
  - Solutions available Friday night-Saturday online on password-protected page
- Final Exam, Monday, June 10, 2:30-4:20 pm MGH 389
  - Topic list and sample final exam problems have been posted
  - Comprehensive final, closed book, closed notes
  - Review session, Sunday, June 9, 4:00 pm EEB 125

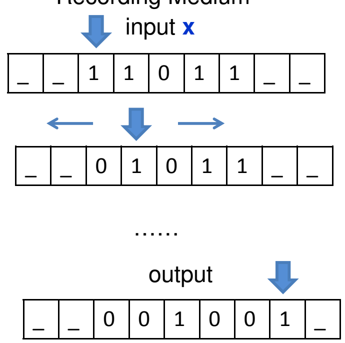
Last lecture highlights

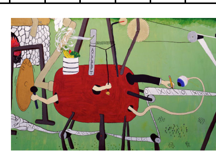
Turing machine = Finite control + Recording Medium + Focus of attention

Finite Control:
program **P**

	_	0	1
s_1	$(1,s_3)$	$(1,s_2)$	$(0,s_2)$
s_2	(H,s_3)	(R,s_1)	(R,s_1)
s_3	(H,s_3)	(R,s_3)	(R,s_3)

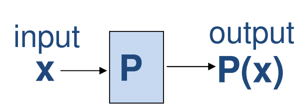
Recording Medium
input **x**

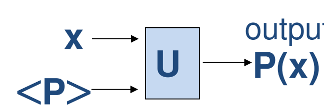




Last lecture highlights

- The Universal Turing Machine **U**
  - Takes as input: **(<P>,x)** where **<P>** is the code of a program and **x** is an input string
  - Simulates **P** on input **x**
- Same as a Program Interpreter

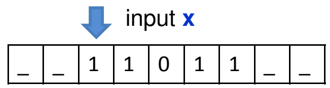


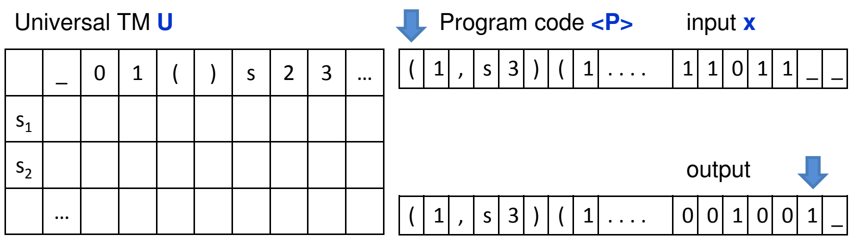


Last lecture highlights

Program **P**

	_	0	1
s_1	$(1,s_3)$	$(1,s_2)$	$(0,s_2)$
s_2	(H,s_3)	(R,s_1)	(R,s_1)
s_3	(H,s_3)	(R,s_3)	(R,s_3)





Programs about Program Properties

- The Universal TM takes a program code **<P>** as input, and an input **x**, and interprets **P** on **x**
  - Step by step by step by step…
- Can we write a TM that takes a program code **<P>** as input and checks some property of the program?
  - Does **P** ever return the output "ERROR"?
  - Does **P** always return the output "ERROR"?
  - Does **P** halt on input **x**?

Halting Problem

- Given: the code of a program **P** and an input **x** for **P**, i.e. given (**<P>**,**x**)
- Output: **1** if **P** halts on input **x**
  0 if **P** does not halt on input **x**

Theorem (Turing): There is no program that solves the halting problem

"The halting problem is undecidable"

Proof by contradiction

- Suppose that **H** is a Turing machine that solves the Halting problem

Function **D**(**x**):
- if **H**(**x**,**x**)=**1** then
  - **while** (true); /* loop forever */
- else
  - **no-op**; /* do nothing and halt */
- endif

- What does **D** do on input **<D>**?
  - Does it halt?

Does **D** halt on input **<D>**?

Function **D(x)**:
- if **H(x,x)=1** then
  - **while** (true); /* loop forever */
- else
  - **no-op**; /* do nothing and halt */
- endif

D halts on input **<D>**

⇔ **H** outputs **1** on input (**<D>**,**<D>**)

[since **H** solves the halting problem and so **H**(**<D>**,**x**) outputs **1** iff **D** halts on input **x**]

⇔ **D** runs forever on input **<D>**

[since **D** goes into an infinite loop on **x** iff **H(x,x)=1**]

That's it!

- We proved that there is no computer program that can solve the Halting Problem.
- This tells us that there is no compiler that can check our programs and guarantee to find any infinite loops they might have

SCOOPING THE LOOP SNOOPER

A proof that the Halting Problem is undecidable

by Geoffrey K. Pullum (U. Edinburgh)

No general procedure for bug checks succeeds.
Now, I won't just assert that, I'll show where it leads:
I will prove that although you might work till you drop,
you cannot tell if computation will stop.

For imagine we have a procedure called *P*
that for specified input permits you to see
whether specified source code, with all of its faults,
defines a routine that eventually halts.

You feed in your program, with suitable data,
and *P* gets to work, and a little while later
(in finite compute time) correctly infers
whether infinite looping behavior occurs...

SCOOPING THE LOOP SNOOPER

...
Here's the trick that I'll use -- and it's simple to do.
I'll define a procedure, which I will call *Q*,
that will use *P*'s predictions of halting success
to stir up a terrible logical mess.
...

And this program called *Q* wouldn't stay on the shelf;
I would ask it to forecast its run on *itself*.
When it reads its own source code, just what will it do?
What's the looping behavior of *Q* run on *Q*?
...

Full poem at:
http://www.lel.ed.ac.uk/~gpullum/loopsnoop.html

Another view of the proof undecidability of the Halting Problem

- Suppose that there is a program **H** that computes the answer to the Halting Problem
- We will build a table with a row for each program (just like we did for uncountability of reals)
- If the supposed program **H** exists then the **D** program we constructed as before will exist and so be in the table
- But **D** must have entries like the "flipped diagonal"
  - **D** can't possibly be in the table.
  - Only assumption was that **H** exists. That must be false.

Some possible inputs **x**

programs **P**

	$\langle P_1 \rangle$	$\langle P_2 \rangle$	$\langle P_3 \rangle$	$\langle P_4 \rangle$	$\langle P_5 \rangle$	$\langle P_6 \rangle$	....					
P_1	0	1	1	0	1	1	1	0	0	0	1	...
P_2	1	1	0	1	0	1	1	0	1	1	1	...
P_3	1	0	1	0	0	0	0	0	0	0	1	...
P_4	0	1	1	0	1	0	1	1	0	1	0	...
P_5	0	1	1	1	1	1	1	0	0	0	1	...
P_6	1	1	0	0	0	1	1	0	1	1	1	...
P_7	1	0	1	1	0	0	0	0	0	0	1	...
P_8	0	1	1	1	1	0	1	1	0	1	0	...
P_9	.	.	.	.	.	.	.	.	.	.	.	
.	.	.	.	.	.	.	.	.	.	.	.	
.												

(**P**,**x**) entry is **1** if program **P** halts on input **x** and **0** if it runs forever

Some possible inputs **x**

programs **P**

	$\langle P_1 \rangle$	$\langle P_2 \rangle$	$\langle P_3 \rangle$	$\langle P_4 \rangle$	$\langle P_5 \rangle$	$\langle P_6 \rangle$	....					
P_1	**0**	1	1	0	1	1	1	0	0	0	1	...
P_2	1	**1**	0	1	0	1	1	0	1	1	1	...
P_3	1	0	**1**	0	0	0	0	0	0	0	1	...
P_4	0	1	1	**0**	1	0	1	1	0	1	0	...
P_5	0	1	1	1	**1**	1	1	0	0	0	1	...
P_6	1	1	0	0	0	**1**	1	0	1	1	1	...
P_7	1	0	1	1	0	0	**0**	0	0	0	1	...
P_8	0	1	1	1	1	0	1	**1**	0	1	0	...
P_9	.	.	.	.	.	.	.	.	.	.	.	
.	.	.	.	.	.	.	.	.	.	.	.	
.												

(**P**,**x**) entry is **1** if program **P** halts on input **x** and **0** if it runs forever

Some possible inputs **x**

D behaves like **flipped diagonal**

programs **P**

	$\langle P_1 \rangle$	$\langle P_2 \rangle$	$\langle P_3 \rangle$	$\langle P_4 \rangle$	$\langle P_5 \rangle$	$\langle P_6 \rangle$	....					
P_1	0 **1**	1	1	0	1	1	1	0	0	0	1	...
P_2	1	1 **0**	0	1	0	1	1	0	1	1	1	...
P_3	1	0	1 **0**	0	0	0	0	0	0	0	1	...
P_4	0	1	1	0 **1**	1	0	1	1	0	1	0	...
P_5	0	1	1	1	1 **0**	1	1	0	0	0	1	...
P_6	1	1	0	0	0	1 **0**	1	0	1	1	1	...
P_7	1	0	1	1	0	0	0 **1**	0	0	0	1	...
P_8	0	1	1	1	1	0	1	1 **0**	0	1	0	...
P_9	.	.	.	.	.	.	.	.	.	.	.	
.	.	.	.	.	.	.	.	.	.	.	.	
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(**P**,**x**) entry is **1** if program **P** halts on input **x** and **0** if it runs forever

Recall: Code for D assuming subroutine H that solves the Halting Problem

- Function **D(x)**:
  - if **H(x,x)=1** then
    - **while** (true); /* loop forever */
  - else
    - **no-op**; /* do nothing and halt */
  - endif

- If **D** existed it would have a row different from every row of the table
  - **D** can't be a program so **H** cannot exist!

Halting Problem

x → H → 1 if P(x) halts
<P> → H → 0 if P(x) does not halt

That's it!

- We proved that there is no computer program that can solve the Halting Problem.

- This tells us that there is no compiler that can check our programs and guarantee to find any infinite loops they might have
  - The full story is even worse

The "Always Halting" problem

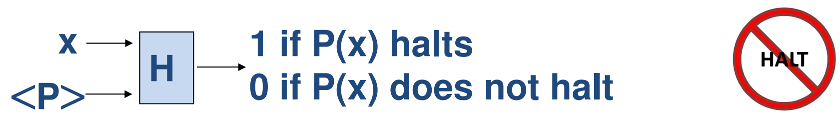
- **Given:** <Q>, the code of a program **Q**
- **Output:** **1** if **Q** halts on every input
  0 if not.

Claim: the "always halts" problem is undecidable

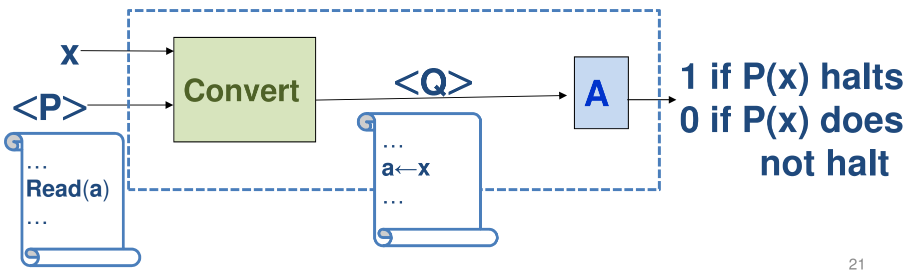
Proof idea:

- Show we could solve the Halting Problem **if** we had a solution for the "always halts" problem.
- No program solving for Halting Problem exists ⇒ no program solving the "always halts" problem exists

The “Always Halting” problem



Suppose we had a TM **A** for the Always Halting problem

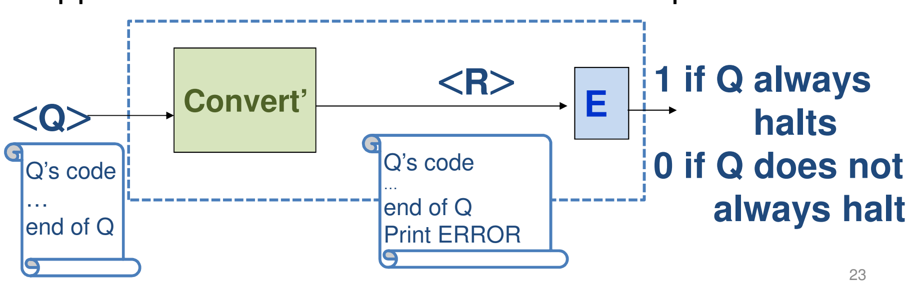


The “Always ERROR” problem

- **Given: <R>,** the code of a program **R**
- **Output: 1** if **R** always prints ERROR
  0 if **R** does not always print ERROR

The “Always ERROR” problem

Suppose we had a TM **E** for the ERROR problem



Pitfalls

- Not *every* problem on programs is undecidable! Which of these is decidable?
- Input <P> and x
  Output: 1 if P prints “ERROR” on x
  after less than 100 steps
  0 otherwise
- Input <P> and x
  Output: 1 if P prints “ERROR” on x
  after more than 100 steps
  0 otherwise