CSE 311 Foundations of Computing I

Lecture 17

Structural Induction

Spring 2013

Announcements

- Reading assignments
  - Today and Monday:
    - 7th Edition, Section 5.3 and pp. 878-880
    - 6th Edition, Section 4.3 and pp. 817-819
- Midterm Friday, May 10, MGH 389
  - Closed book, closed notes
  - Tables of inference rules and equivalences will be included on test
  - Sample questions from old midterms are now posted

Highlight from last time… Recursive Definitions of Set S

- Recursive definition
  - *Basis step:* Some specific elements are in S
  - *Recursive step:* Given some existing named elements in S some new objects constructed from these named elements are also in S.
  - Exclusion rule: Every element in S follows from basis steps and a finite number of recursive steps

Highlight from last time… Strings

- An *alphabet* Σ is any finite set of characters.
- The set Σ^* of *strings* over the alphabet Σ is defined by
  - **Basis:** $\lambda \in \Sigma^*$ (λ is the empty string)
  - **Recursive:** if $w \in \Sigma^*$, $a \in \Sigma$, then $wa \in \Sigma^*$

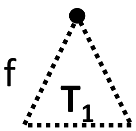
Highlight from last time…
Functions on recursively defined sets

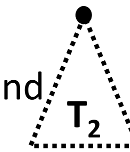
len (λ) = 0;
len (wa) = 1 + len(w); for $w \in \Sigma^*$, $a \in \Sigma$

Reversal:
$\lambda^R = \lambda$
$(wa)^R = aw^R$ for $w \in \Sigma^*$, $a \in \Sigma$

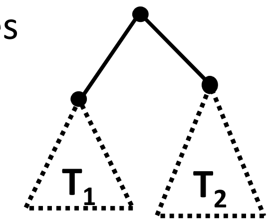
Concatenation:
$x \bullet \lambda = x$ for $x \in \Sigma^*$
$x \bullet wa = (x \bullet w)a$ for $x, w \in \Sigma^*$, $a \in \Sigma$

Highlight from last time…
Rooted Binary trees

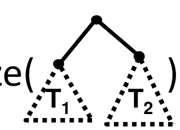
- **Basis:** • is a rooted binary tree
- **Recursive Step:** If 

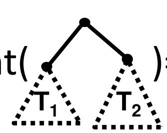
and 

are rooted binary trees then so is:



Highlight from last time…
Functions defined on rooted binary trees

- size(•)=1
- size(

) = 1+size(T_1)+size(T_2)
- height(•)=0
- height(

)=1+max{height(T_1),height(T_2)}

Structural Induction: Proving properties of recursively defined sets

How to prove $\forall x \in S.\ P(x)$ is true:

1. Let P(x) be "…". We will prove P(x) for all x∈S
2. **Base Case:** Show that P is true for all specific elements of S mentioned in the *Basis step*
3. **Inductive Hypothesis:** Assume that P is true for some arbitrary values of each of the existing named elements mentioned in the *Recursive step*
4. **Inductive Step:** Prove that P holds for each of the new elements constructed in the *Recursive step* using the named elements mentioned in the Inductive Hypothesis
5. Conclude that $\forall x \in S.\ P(x)$

Structural Induction versus Ordinary Induction

- Ordinary induction is a special case of structural induction:
  - Recursive Definition of $\mathbb{N}$
    - **Basis:** $0 \in \mathbb{N}$
    - **Recursive Step:** If $k \in \mathbb{N}$ then $k+1 \in \mathbb{N}$
- Structural induction follows from ordinary induction
  - Let Q(n) be true iff for all $x \in S$ that take n Recursive steps to be constructed, P(x) is true.

Using Structural Induction

- Let S be given by
  - **Basis:** $6 \in S$; $15 \in S$;
  - **Recursive:** if $x, y \in S$, then $x + y \in S$.
- **Claim:** Every element of S is divisible by 3

Using Structural Induction

- Let S be a set of strings over {a,b} defined as follows
  - Basis: $a \in S$
  - Recursive:
    - If $u \in S$ then $au \in S$ and $bau \in S$
    - If $u \in S$ and $v \in S$ then $uv \in S$
- Claim: if $x \in S$ then x has more a's than b's

len(x•y)=len(x)+len(y) for all strings x and y

Let P(w) be "For all strings x, len(x•w)=len(x)+len(w)"

For every rooted binary tree T size(T) $\leq 2^{\mathbf{height(T)+1}} - 1$

Languages: Sets of Strings

- Sets of strings that satisfy special properties are called *languages*. Examples:
  - English sentences
  - Syntactically correct Java/C/C++ programs
  - All strings over alphabet Σ
  - Palindromes over Σ
  - Binary strings that don't have a 0 after a 1
  - Legal variable names. keywords in Java/C/C++
  - Binary strings with an equal # of 0's and 1's

Regular Expressions over Σ

- Each is a "pattern" that specifies a set of strings
- Basis:
  - $\varnothing$, λ are regular expressions
  - ***a*** is a regular expression for any $a \in \Sigma$
- Recursive step:
  - If **A** and **B** are regular expressions then so are:
    - (**A** $\cup$ **B**)
    - (**AB**)
    - **A***

Each regular expression is a "pattern"

- λ matches the empty string
- ***a*** matches the one character string *a*
- (**A** $\cup$ **B**) matches all strings that either **A** matches or **B** matches (or both)
- (**AB**) matches all strings that have a first part that **A** matches followed by a second part that **B** matches
- **A*** matches all strings that have any number of strings (even 0) that **A** matches, one after another

Examples

- 0^*
- 0^*1^*
- $(0 \cup 1)^*$
- $(0^*1^*)^*$
- $(0 \cup 1)^*\,0110\,(0 \cup 1)^*$
- $(0 \cup 1)^*\,(0110 \cup 100)(0 \cup 1)^*$

Regular expressions in practice

- Used to define the "tokens": e.g., legal variable names, keywords in programming languages and compilers
- Used in **`grep`**, a program that does pattern matching searches in UNIX/LINUX
- Pattern matching using regular expressions is an essential feature of hypertext scripting language PHP used for web programming
  - Also in text processing programming language Perl

Regular Expressions in PHP

- int **preg_match** (string $pattern , string $subject,...)
- $pattern syntax:

`[01]`	a 0 or a 1	`^` start of string	`$` end of string
`[0-9]`	any single digit	`\.` period	`\,` comma `\-` minus
`.`	any single character		
ab	a followed by b	(**AB**)	
`(a|b)`	a or b	($\mathbf{A} \cup \mathbf{B}$)	
a**?**	zero or one of a	($\mathbf{A} \cup \lambda$)	
a*****	zero or more of a	$\mathbf{A}^*$	
a**+**	one or more of a	$\mathbf{AA}^*$	

- e.g. **`^[\-+]?[0-9]*(\.|\,)?[0-9]+$`**

  General form of decimal number e.g. 9.12 or -9,8 (Europe)

More examples

- All binary strings that have an even # of 1's
- All binary strings that *don't* contain 101

Regular expressions can't specify everything we might want

- **Fact**: Not all sets of strings can be specified by regular expressions
  - One example is the set of binary strings with equal #'s of 0's and 1's