CSE 311 Foundations of Computing I

Lecture 14

Induction and Strong Induction

Spring 2013

Announcements

- Reading assignments
  - Today:
    - 5.1-5.2 7th Edition
    - 4.1-4.2 6th Edition
    - 3.3, 3.4 5th Edition
- Homework 4 due today, Homework 3 back
- Homework 5 out today, due May 8
- Midterm Friday, May 10, MGH 389
  - Closed book, closed notes
  - Tables of inference rules and equivalences will be included on test

Highlights from last lecture

- Mathematical Induction

$$\begin{array}{l} P(0) \\ \underline{\forall k\geq 0\ (P(k) \rightarrow P(k+1))} \\ \therefore \forall n\geq 0\ \ P(n) \end{array}$$

- Induction proof layout:
  1. By induction we will show that P(n) is true for every n≥0
  2. Base Case: Prove P(0)
  3. Inductive Hypothesis: Assume that P(k) is true for some arbitrary integer $k \geq 0$
  4. Inductive Step: Prove that P(k+1) is true using Inductive Hypothesis that P(k) is true
  5. Conclusion: Result follows by induction

Highlights from last lecture: Claim: $3 \mid 2^{2n} - 1$ for all $n \geq 0$

Proof:

1. Let P(n) be "$3 \mid 2^{2n} - 1$". We will prove by induction that P(n) is true for all integers $n \geq 0$.
2. <u>Base Case</u>: n=0. $2^{2\bullet 0}-1=2^0-1=1-1=0=3\bullet 0$. Therefore $3 \mid 2^{2\bullet 0}-1$ so P(0) is true ✓
3. <u>Inductive Hypothesis</u>: Assume that $3 \mid 2^{2k} - 1$ for some arbitrary integer $k \geq 0$.
4. <u>Inductive Step</u>: $\boxed{\text{Goal: Show } 3 \mid 2^{2(k+1)} - 1}$

Highlights from last lecture: Claim: $3 \mid 2^{2n} - 1$ for all $n \geq 0$

Proof continued…

4. Inductive Step: $\boxed{\text{Goal: Show } 3 \mid 2^{2(k+1)} - 1}$

   By Inductive Hypothesis there is some integer m such that $2^{2k} - 1 = 3m$.

   Now $2^{2(k+1)} - 1 = 2^{2k+2} - 1 = 4 \bullet 2^{2k} - 1 = 4(3m+1) - 1$
   $= 12m + 3 = 3(4m+1)$

   Since $4m+1$ is an integer, $3 \mid 2^{2(k+1)} - 1$ ✓

5. Conclusion: Therefore, by induction we have proved that $3 \mid 2^{2n} - 1$ for all $n \geq 0$

$1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$ for all $n \geq 0$

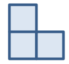
$1+2+\ldots+n = \sum_{i=1}^{n} i = n(n+1)/2$ for all $n \geq 1$

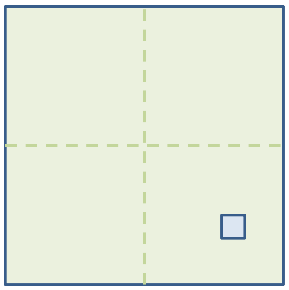
Harmonic Numbers

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \frac{1}{n} = \sum_{k=1}^{n} \frac{1}{k}$$

Prove $H_{2^n} \geq 1 + \frac{n}{2}$ for all $n \geq 1$

Cute Application: Checkerboard tiling with Tri-ominos

Prove that a $2^n \times 2^n$ checkerboard with one square removed can be tiled with: 



Strong Induction

$P(0)$
$\forall\, k\ ((P(0) \wedge P(1) \wedge P(2) \wedge \ldots \wedge P(k)) \rightarrow P(k+1))$
$\therefore\ \forall\, n\ P(n)$

Follows from ordinary induction applied to
$Q(n) = P(0) \wedge P(1) \wedge P(2) \wedge \ldots \wedge P(n)$

Strong Induction English Proofs

1. By induction we will show that P(n) is true for every n≥0
2. Base Case: Prove P(0)
3. Inductive Hypothesis:
   Assume that for some arbitrary integer k ≥ 0, P(j) is true for every j from 0 to k
4. Inductive Step:
   Prove that P(k+1) is true using the Inductive Hypothesis (that P(j) is true for all values ≤ k)
5. Conclusion: Result follows by induction

Every integer ≥ 2 is the product of primes

Recursive Definitions of Functions

- $F(0) = 0$; $F(n + 1) = F(n) + 1$ for all $n \geq 0$
- $G(0) = 1$; $G(n + 1) = 2 \times G(n)$ for all $n \geq 0$
- $0! = 1$; $(n+1)! = (n+1) \times n!$ for all $n \geq 0$
- $H(0) = 1$; $H(n + 1) = 2^{H(n)}$ for all $n \geq 0$

Fibonacci Numbers

- $f_0 = 0$; $f_1 = 1$; $f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

Bounding the Fibonacci Numbers

- Theorem: $2^{n/2-1} \leq f_n < 2^n$ for all $n \geq 2$

Fibonacci numbers and the running time of Euclid's algorithm

- Theorem: Suppose that Euclid's algorithm takes n steps for gcd(a,b) with a>b, then $a \geq f_{n+1}$
- Set $r_{n+1}=a$, $r_n=b$ then Euclid's alg. computes

$$
\begin{aligned}
r_{n+1} &= q_n r_n + r_{n-1} \\
r_n &= q_{n-1} r_{n-1} + r_{n-2} \\
&\vdots \\
r_3 &= q_2 r_2 + r_1 \\
r_2 &= q_1 r_1
\end{aligned}
$$

each quotient $q_i \geq 1$
$r_1 \geq 1$