CSE 311 Foundations of Computing I

Lecture 7

Proofs

Spring 2013

Announcements

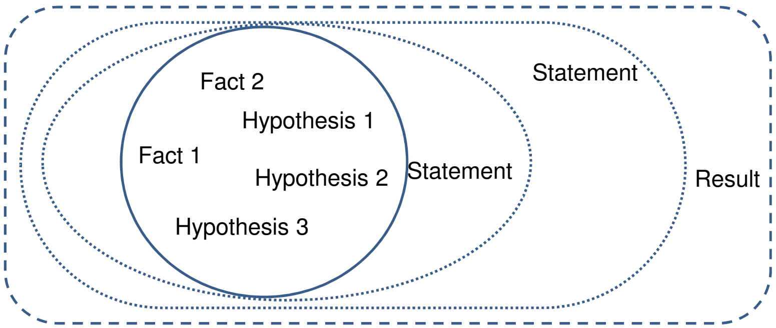
- Reading assignments
  - Logical Inference
    - 1.6, 1.7 7th Edition
    - 1.5, 1.6 6th Edition
- Homework
  - Graded HW 1 available starting in Tuesday's office hours
  - HW 2 due Wednesday

Highlights from last lecture

- Predicate logic, intricacies of $\forall$, $\exists$
- Introduction to inference…

Highlights…Proofs

- Start with hypotheses and facts
- Use rules of inference to extend set of facts
- Result is proved when it is included in the set



Review…An inference rule: *Modus Ponens*

- If p and p→q are both true then q must be true
- Write this rule as $\frac{p,\ p \to q}{\therefore\ q}$
- Given:
  - If it is Monday then you have 311 homework due today.
  - It is Monday.
- Therefore, by Modus Ponens:
  - You have 311 homework due today.

Review…Proofs

- Show that r follows from p , p→q, and q→r

1. p Given
2. p→q Given
3. q →r Given
4. q Modus Ponens from 1 and 2
5. r Modus Ponens from 3 and 4

Review…Proofs can use Equivalences too

Show that ¬p follows from p→q and ¬q

1. p→q Given
2. ¬q Given
3. ¬q → ¬p Contrapositive of 1 (Equivalence!)
4. ¬p Modus Ponens from 2 and 3

Review…Important: Applications of Inference Rules

- You can use equivalences to make substitutions of any subformula
- Inference rules only can be applied to whole formulas (not correct otherwise).

e.g. 1. p→q Given

~~2. (p ∨ r) →q Intro ∨ from 1.~~

Does not follow! e.g p=**F**, q=**F**, r=**T**

Review…Inference Rules

- Each *inference rule* is written as $\dfrac{A, B}{\therefore C, D}$ which means that if both A and B are true then you can infer C and you can infer D.
  - For rule to be correct $(A \wedge B) \rightarrow C$ and $(A \wedge B) \rightarrow D$ must be a tautologies
- Sometimes rules don't need anything to start with. These rules are called *axioms*:
  - e.g. *Excluded Middle Axiom* $\dfrac{\quad}{\therefore\ p \vee \neg p}$

Review…Simple Propositional Inference Rules

- Excluded middle plus two inference rules per binary connective, one to eliminate it and one to introduce it

$$\frac{p \wedge q}{\therefore\ p, q} \qquad \frac{p, q}{\therefore\ p \wedge q} \qquad \frac{\quad}{\therefore\ p \vee \neg p}$$

$$\frac{p \vee q\ ,\ \neg p}{\therefore\ q} \qquad \frac{p}{\therefore\ p \vee q,\ q \vee p}$$

$$\frac{p,\ p \rightarrow q}{\therefore\ q} \qquad \frac{p \Rightarrow q}{\therefore\ p \rightarrow q}$$

Direct Proof Rule

Not like other rules!

See next slide…

Direct Proof of an Implication

- $p \Rightarrow q$ denotes a proof of q given p as an assumption. Don't confuse with $p \rightarrow q$.
- The direct proof rule
  - if you have such a proof then you can conclude that $p \rightarrow q$ is true
- E.g. Let's prove $p \rightarrow (p \vee q)$

1.	p	Assumption
2.	$p \vee q$	Intro for $\vee$ from 1
3.	$p \rightarrow (p \vee q)$	Direct proof rule

Proof subroutine for $p \Rightarrow (p \vee q)$

Proofs using Direct Proof Rule

- Show that $p \rightarrow r$ follows from q and $(p \wedge q) \rightarrow r$

1.	q	Given
2.	$(p \wedge q) \rightarrow r$	Given
3.	p	Assumption
4.	$p \wedge q$	From 1 and 3 via Intro $\wedge$ rule
5.	r	Modus Ponens from 2 and 4
6.	$p \rightarrow r$	Direct Proof rule

Example

- Prove $((p \to q) \wedge (q \to r)) \to (p \to r)$

One General Proof Strategy

1. Look at the rules for introducing connectives to see how you would build up the formula you want to prove from pieces of what is given
2. Use the rules for eliminating connectives to break down the given formulas so that you get the pieces you need to do 1.
3. Write the proof beginning with what you figured out for 2 followed by 1.

Inference Rules for Quantifiers

$$\frac{P(c) \text{ for some } c}{\therefore \exists x\, P(x)} \qquad \frac{\forall x\, P(x)}{\therefore P(a) \text{ for any } a}$$

$$\frac{\text{“Let } a \text{ be anything*”} \ldots P(a)}{\therefore \forall x\, P(x)} \qquad \frac{\exists x\, P(x)}{\therefore P(c) \text{ for some special } c}$$

* in the domain of P

Proofs using Quantifiers

“There exists an even prime number”

Prime(*x*): x is an integer > 1 and x is not a multiple of any integer strictly between 1 and x

Even and Odd

$\text{Even}(x) \equiv \exists y\ (x=2y)$
$\text{Odd}(x) \equiv \exists y\ (x=2y+1)$
Domain: Integers

Prove: "The square of every even number is even"

Formal proof of: $\forall x\ (\text{Even}(x) \rightarrow \text{Even}(x^2))$

Even and Odd

$\text{Even}(x) \equiv \exists y\ (x=2y)$
$\text{Odd}(x) \equiv \exists y\ (x=2y+1)$
Domain: Integers

Prove: "The square of every odd number is odd"

English proof of: $\forall x\ (\text{Odd}(x) \rightarrow \text{Odd}(x^2))$

Let x be an odd number.

Then $x=2k+1$ for some integer k (depending on x)

Therefore $x^2=(2k+1)^2= 4k^2+4k+1=2(2k^2+2k)+1$.

Since $2k^2+2k$ is an integer, x^2 is odd.

"Proof by Contradiction": One way to prove $\neg p$

If we assume p and derive False (a contradiction) then we have proved $\neg p$.

1. p Assumption

...

3. **F**

4. $p \rightarrow$ **F** Direct Proof rule
5. $\neg p \vee$ **F** Equivalence from 4
6. $\neg p$ Equivalence from 5

Even and Odd

$\text{Even}(x) \equiv \exists y\ (x=2y)$
$\text{Odd}(x) \equiv \exists y\ (x=2y+1)$
Domain: Integers

Prove: "No number is both even and odd"

English proof: $\neg \exists x\ (\text{Even}(x) \wedge \text{Odd}(x))$

$\equiv \forall x\ \neg(\text{Even}(x) \wedge \text{Odd}(x))$

Let x be any integer and suppose that it is both even and odd. Then $x=2k$ for some integer k and $x=2n+1$ for some integer n. Therefore $2k=2n+1$ and hence $k=n+½$.

But two integers cannot differ by ½ so this is a contradiction.

Rational Numbers

- A real number x is *rational* iff there exist integers p and q with $q \neq 0$ such that $x=p/q$.

$\text{Rational}(x) \equiv \exists p\, \exists q\ ((x=p/q) \wedge \text{Integer}(p) \wedge \text{Integer}(q) \wedge q \neq 0)$

- Prove:
  - If x and y are rational then xy is rational

$\forall x\, \forall y\, ((\text{Rational}(x) \wedge \text{Rational}(y)) \rightarrow \text{Rational}(xy))$

Domain: Real numbers

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  - If x and y are rational then x+y is rational

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  - If x and y are rational then x+y is rational
  - If x and y are rational then x/y is rational

Counterexamples

- To *disprove* $\forall x\, P(x)$ find a *counterexample*
  - some c such that $\neg P(c)$
  - works because this implies $\exists x\, \neg P(x)$ which is equivalent to $\neg \forall x\, P(x)$

Proofs

- Formal proofs follow simple well-defined rules and should be easy to check
  - In the same way that code should be easy to execute
- English proofs correspond to those rules but are designed to be easier for humans to read
  - Easily checkable in principle
- Simple proof strategies already do a lot
  - Later we will cover a specific strategy that applies to loops and recursion (mathematical induction)