CSE 311 Foundations of Computing I

Lecture 6

Predicate Logic, Logical Inference

Spring 2013

Announcements

- Reading assignments
  - Logical Inference
    - 1.6, 1.7 7th Edition
    - 1.5, 1.6, 1.7 6th Edition

Highlights from last lecture

- Predicates
  - Cat(x), Prime(x), HasTaken(s,c)
- Quantifiers
  - $\forall x\,(\mathit{Even}(x) \vee \mathit{Odd}(x))$, $\exists x\,(\mathrm{Cat}(x) \wedge \mathrm{LikesTofu}(x))$
- Correspondence between English and logic
  - "Red cats like tofu"
  - $\forall x\,((\mathrm{Cat}(x) \wedge \mathrm{Red}(x)) \rightarrow \mathrm{LikesTofu}(x))$
- Nested quantifiers
  - $\forall x\,\exists y\,\mathrm{Greater}\,(y, x)$

Highlights from Last Lecture
Scope of Quantifiers

- $\mathrm{Notlargest}(x) \equiv \exists y\,\mathrm{Greater}\,(y, x)$
  $\equiv \exists z\,\mathrm{Greater}\,(z, x)$
  - Value doesn't depend on y or z "bound variables"
  - Value does depend on x "free variable"
- Quantifiers only act on free variables of the formula they quantify
  - $\forall x\,(\exists y\,(P(x,y) \rightarrow \forall x\,Q(y, x)))$

Nested Quantifiers

- Bound variable name doesn't matter
  - $\forall x \exists y\, P(x, y) \equiv \forall a \exists b\, P(a, b)$
- Positions of quantifiers can sometimes change
  - $\forall x\, (Q(x) \wedge \exists y\, P(x, y)) \equiv \forall x \exists y\, (Q(x) \wedge P(x, y))$
- BUT: Order is important...

Quantification with two variables

Expression	When true	When false
$\forall x \forall y\, P(x, y)$		
$\exists x \exists y\, P(x, y)$		
$\forall x \exists y\, P(x, y)$		
$\exists y \forall x\, P(x, y)$		

Negations of Quantifiers

- Not every positive integer is prime
- Some positive integer is not prime
- Prime numbers do not exist
- Every positive integer is not prime

De Morgan's Laws for Quantifiers

$$\boxed{\begin{array}{c} \neg\forall x\ P(x) \equiv \exists x\, \neg P(x) \\ \neg\exists x\, P(x) \equiv \forall x\, \neg P(x) \end{array}}$$

De Morgan's Laws for Quantifiers

$$\neg\forall x\ \ P(x) \equiv \exists x\ \neg P(x)$$
$$\neg\ \exists x\ P(x) \equiv \forall x\ \neg P(x)$$

"There is no largest integer"

$$\neg\ \exists\ x\ \forall\ y\ \ (\ x \geq y)$$
$$\equiv \forall\ x\ \neg\ \forall y\ \ (\ x \geq y)$$
$$\equiv \forall\ x\ \ \exists\ y\ \neg\ (\ x \geq y)$$
$$\equiv \forall\ x\ \ \exists\ y\ \ \ \ (y > x)$$

"For every integer there is a larger integer"

Logical Inference

- So far we've considered
  - How to understand and *express* things using propositional and predicate logic
  - How to *compute* using Boolean (propositional) logic
  - How to show that different ways of expressing or computing them are *equivalent* to each other
- Logic also has methods that let us *infer* implied properties from ones that we know
  - Equivalence is a small part of this

Applications of Logical Inference

- Software Engineering
  - Express desired properties of program as set of logical constraints
  - Use inference rules to show that program implies that those constraints are satisfied
- AI
  - Automated reasoning
- Algorithm design and analysis
  - e.g., Correctness, Loop invariants.
- Logic Programming, e.g. Prolog
  - Express desired outcome as set of constraints
  - Automatically apply logic inference to derive solution

Proofs

- Start with hypotheses and facts
- Use rules of inference to extend set of facts
- Result is proved when it is included in the set

An inference rule: *Modus Ponens*

- If p and $p \rightarrow q$ are both true then q must be true
- Write this rule as $\dfrac{p,\ p \rightarrow q}{\therefore\ q}$
- Given:
  - If it is Friday then you have a 311 class today.
  - It is Friday.
- Therefore, by Modus Ponens:
  - You have a 311 class today

Proofs

- Show that r follows from p , $p \rightarrow q$, and $q \rightarrow r$

1. p Given
2. $p \rightarrow q$ Given
3. $q \rightarrow r$ Given
4. q Modus Ponens from 1 and 2
5. r Modus Ponens from 3 and 4

Proofs can use Equivalences too

Show that $\neg p$ follows from $p \rightarrow q$ and $\neg q$

1. $p \rightarrow q$ Given
2. $\neg q$ Given
3. $\neg q \rightarrow \neg p$ Contrapositive of 1
4. $\neg p$ Modus Ponens from 2 and 3

Inference Rules

- Each *inference rule* is written as $\dfrac{A,\ B}{\therefore\ C, D}$ which means that if both A and B are true then you can infer C and you can infer D.
  - For rule to be correct $(A \wedge B) \rightarrow C$ and $(A \wedge B) \rightarrow D$ must be a tautologies
- Sometimes rules don't need anything to start with. These rules are called *axioms*:
  - e.g. *Excluded Middle Axiom* $\dfrac{\qquad}{\therefore\ p \vee \neg p}$

Important: Applications of Inference Rules

- You can use equivalences to make substitutions of any subformula
- Inference rules only can be applied to whole formulas (not correct otherwise).

e.g. 1. $p \to q$ Given

~~2. $(p \lor r) \to q$ Intro $\lor$ from 1.~~

Does not follow! e.g $p=$**F**, $q=$**F**, $r=$**T**

Simple Propositional Inference Rules

- Excluded middle plus two inference rules per binary connective, one to eliminate it and one to introduce it

$$\frac{p \land q}{\therefore p,\ q} \qquad \frac{p,\ q}{\therefore p \land q}$$

$$\frac{p \lor q,\ \neg p}{\therefore q} \qquad \frac{p}{\therefore p \lor q,\ q \lor p}$$

$$\frac{p,\ p \to q}{\therefore q} \qquad \frac{p \Rightarrow q}{\therefore p \to q}$$

Direct Proof Rule

Not like other rules

Direct Proof of an Implication

- $p \Rightarrow q$ denotes a proof of q given p as an assumption
- The direct proof rule
  - if you have such a proof then you can conclude that $p \to q$ is true
- E.g.

Proof subroutine

1. p Assumption
2. $p \lor q$ Intro for $\lor$ from 1
3. $p \to (p \lor q)$ Direct proof rule

Inference Rules for Quantifiers

$$\frac{P(c) \text{ for some } c}{\therefore \exists\, x\, P(x)} \qquad \frac{\forall\, x\, P(x)}{\therefore P(a) \text{ for any } a}$$

$$\frac{\text{“Let a be anything”...}P(a)}{\therefore \forall\, x\, P(x)} \qquad \frac{\exists\, x\, P(x)}{\therefore P(c) \text{ for some special } c}$$

Proofs using Quantifiers

- Show that "Simba is a cat" follows from "All lions are cats" and "Simba is a lion" (using the domain of all animals)

Proofs using Quantifiers

- "There exists an even prime number"

General Proof Strategy

A. Look at the rules for introducing connectives to see how you would build up the formula you want to prove from pieces of what is given
B. Use the rules for eliminating connectives to break down the given formulas so that you get the pieces you need to do A.
C. Write the proof beginning with what you figured out for B followed by A.