

Sample midterm questions

Instructions:

- Exam will consist of 5 to 8 questions.
- Closed book, closed notes, no cell phones, no calculators.
- Time limit: 50 minutes.
- Answer the problems on the exam paper.
- If you need extra space use the back of a page.
- Lists of equivalences and inference rules for your use are given on the final two pages.

Problem 1:

- Show that the expression $(p \rightarrow q) \rightarrow (p \rightarrow r)$ is a contingency.
- Give an expression that is logically equivalent to $(p \rightarrow q) \rightarrow (p \rightarrow r)$ using the logical operators \neg , \vee , and \wedge (but not \rightarrow).

Problem 2:

Using the predicates:

$Likes(p, f)$: "Person p likes to eat the food f ."

$Serves(r, f)$: "Restaurant r serves the food f ."

translate the following statements into logical expressions.

- Every restaurant serves a food that no one likes.
- Every restaurant that serves TOFU also serves a food which RANDY does not like.

Problem 3:

Use rules of inference to show that if the premises $\forall x(P(x) \rightarrow Q(x))$, $\forall x(Q(x) \rightarrow R(x))$, and $\neg R(a)$, where a is in the domain, are true, then the conclusion $\neg P(a)$ is true. (Note: You do not need to give the names for the rules of inference.)

Problem 4:

Prove that if n is even and m is odd, then $(n + 1)(m + 1)$ is even.

Problem 5:

Prove or disprove:

- a) For positive integers x , p , and q , $(x \bmod p) \bmod q = x \bmod pq$.
- b) For positive integers x , p , and q , $(x \bmod p) \bmod q = (x \bmod q) \bmod p$.

Problem 6:

- a) Find the multiplicative inverse of 2 modulo 9 (in other words, find a solution to the equation $2x \bmod 9 = 1$.)
- b) Which integers in $\{1, 2, \dots, 8\}$ have multiplicative inverses modulo 9?

Problem 7:

Let $T(n)$ be defined by: $T(0) = 1$, $T(n) = 2nT(n - 1)$ for $n \geq 1$. Prove that for all $n \geq 0$, $T(n) = 2^n n!$.

Problem 8:

Let x_1, x_2, \dots, x_n be odd integers. Prove by induction that $x_1 x_2 \cdots x_n$ is also an odd integer.

Problem 9:

Determine whether the following compound proposition is a tautology, a contradiction, or a contingency: $((s \vee p) \wedge (s \vee \neg p)) \rightarrow ((p \rightarrow q) \rightarrow r)$.

Problem 10:

Find predicates $P(x)$ and $Q(x)$ such that $\forall x(P(x) \oplus Q(x))$ is true, but $\forall xP(x) \oplus \forall xQ(x)$ is false.

Problem 11:

Show that the following is a tautology: $((\neg p \vee q) \wedge (p \vee r)) \rightarrow (q \vee r)$.

Problem 12:

Prove that the sum of an odd number and an even number is an odd number.

Problem 13:

Use mathematical induction to show that 3 divides $n^3 - n$ whenever n is a non-negative integer.

Problem 14:

Let the predicates $D(x, y)$ mean “team x defeated team y ” and $P(x, y)$ mean “team x has played team y .” Give quantified formulas with the following meanings:

- a) Every team has lost at least one game.
- b) There is a team that has beaten every team it has played.

Problem 15:

Let $P(x, y)$ be the predicate “ $x < y$ ” and let the universe for all variables be the real numbers. Express each of the following statements as predicate logic formulas using P :

- a) For every number there is a smaller one.
- b) 7 is smaller than any other number.
- c) 7 is between a and b . (Don't forget to handle both the possibility that b is smaller than a as well as the possibility that a is smaller than b .)
- d) Between any two different numbers there is another number.
- e) For any two numbers, if they are different then one is less than the other.

Problem 16:

Let $V(x, y)$ be the predicate “ x voted for y ”, let $M(x, y)$ be the predicate “ x received more votes than y ”, and let the universe for all variables be the set of all people. Express each of the following statements as predicate logic formulas using V and M :

- a) Everybody received at least one vote.
- b) Jane and John voted for the same person.
- c) Ross won the election. (The winner is the person who received the most votes.)
- d) Nobody who votes for him/herself can win the election.
- e) Everybody can vote for at most one person.

Problem 17:

Prove the following for all natural numbers n by induction, $\sum_{i=0}^n \frac{i}{2^i} = 2 - \frac{n+2}{2^n}$.

Problem 18:

Use Euclid's algorithm to help you solve $11x \equiv 4 \pmod{27}$ for x .

Problem 19:

Write an expression equivalent to $(p \rightarrow q) \rightarrow r$ that is:

- a) A sum of products
- b) A product of sums

Equivalences	
Identity Laws	$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$
Domination Laws	$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$
Idempotent Laws	$p \vee p \equiv p$ $p \wedge p \equiv p$
Commutative Laws	$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$
Associative Laws	$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
Distributive Laws	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
De Morgan's Laws	$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$
Negation Laws	$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$
Double Negation Law	$\neg\neg p \equiv p$
Contrapositive Law	$p \rightarrow q \equiv \neg q \rightarrow \neg p$
Implication Law	$p \rightarrow q \equiv \neg p \vee q$
Quantifier Negation Laws	$\neg\exists xP(x) \equiv \forall x\neg P(x)$ $\neg\forall xP(x) \equiv \exists x\neg P(x)$

Propositional and Predicate Equivalences

Inferences	
Modus Ponens	$\frac{p, p \rightarrow q}{\therefore q}$
Direct Proof	$\frac{p \Rightarrow q}{\therefore p \rightarrow q}$
Elim \wedge	$\frac{p \wedge q}{\therefore p, q}$
Intro \wedge	$\frac{p, q}{\therefore p \wedge q}$
Elim \vee	$\frac{p \vee q, \neg p}{\therefore q}$
Intro \vee	$\frac{p}{\therefore p \vee q, q \vee p}$
Excluded Middle	$\frac{}{\therefore p \vee \neg p}$
Elim \forall	$\frac{\forall x P(x)}{\therefore P(a) \text{ for any } a}$
Intro \forall	$\frac{\text{Let } a \text{ be anything...} P(a)}{\therefore \forall x P(x)}$
Elim \exists	$\frac{\exists x P(x)}{\therefore P(c) \text{ for some special } c}$
Intro \exists	$\frac{P(c) \text{ for some } c}{\therefore \exists x P(x)}$

Propositional and Predicate Inferences