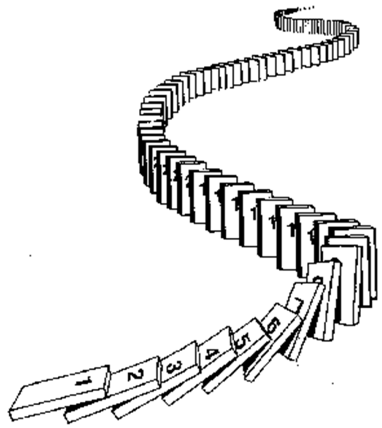
# CSE 311: Foundations of Computing

**Fall 2013**

**Lecture 15: Strong Induction & Recursion**



## announcements

**Reading assignment**

**Induction**

5.1-5.2, 7th edition

4.1-4.2, 6th edition

## review: using the induction rule in a formal proof

$$\frac{\begin{array}{l} P(0) \\ \forall\, k\, (P(k) \rightarrow P(k+1)) \end{array}}{\therefore\ \forall\, n\, P(n)}$$

**Base Case**
1. Prove P(0)

**Inductive Hypothesis**
2. Let k be an arbitrary integer ≥ 0
3. Assume that P(k) is true

**Inductive Step**
4. ...
5. Prove P(k+1) is true

**Conclusion**
6. P(k) → P(k+1) — Direct Proof Rule
7. ∀ k (P(k) → P(k+1)) — Intro ∀ from 2-6
8. ∀ n P(n) — Induction Rule 1&7

## review: 5 steps to inductive proofs in english

**Proof:**

1. "By induction we will show that P(n) is true for every n≥0."
2. "Base Case:" Prove P(0)
3. "Inductive Hypothesis:"
   Assume P(k) is true for some arbitrary integer k ≥ 0"
4. "Inductive Step:" Want to prove that P(k+1) is true:
   Use the goal to figure out what you need.
   Make sure you are using I.H. and point out where you are using it. (Don't assume P(k+1) !!)
5. "Conclusion: Result follows by induction"

## harmonic numbers

$$H_m = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{m} = \sum_{i=1}^{m} \frac{1}{i}$$

**Prove that** $H_{2^n} \geq 1 + \frac{n}{2}$ **for all** $n \geq 1$.

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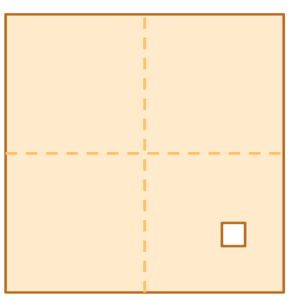
## checkerboard tiling

**Prove that a $2^n \times 2^n$ checkerboard with one square removed can be tiled with:**



## strong induction

$P(0)$

$\forall k \ \big( \big(P(0) \wedge P(1) \wedge P(2) \wedge \cdots \wedge P(k)\big) \rightarrow P(k+1) \big)$

$\therefore \forall n \ P(n)$

**Follows from ordinary induction applied to**
$Q(n) = P(0) \wedge P(1) \wedge P(2) \wedge \cdots \wedge P(n)$

## strong induction english proofs

1. By induction we will show that $P(n)$ is true for every $n \geq 0$
2. Base Case: Prove $P(0)$
3. Inductive Hypothesis:
   Assume that for some arbitrary integer $k \geq 0$, $P(j)$ is true for every $j$ from $0$ to $k$
4. Inductive Step:
   Prove that $P(k+1)$ is true using the Inductive Hypothesis (that $P(j)$ is true for all values $\leq k$)
5. Conclusion: Result follows by induction

## every integer $\geq 2$ is the product of primes

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## recursive definitions of functions

- $F(0) = 0$; $F(n+1) = F(n) + 1$ for all $n \geq 0$
- $G(0) = 1$; $G(n+1) = 2 \times G(n)$ for all $n \geq 0$
- $0! = 1$; $(n+1)! = (n+1) \times n!$ for all $n \geq 0$
- $H(0) = 1$; $H(n+1) = 2^{H(n)}$ for all $n \geq 0$

## fibonacci numbers

$f_0 = 0$
$f_1 = 1$
$f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

## bounding the fibonacci numbers

**Theorem:** $2^{\frac{n}{2}-1} \leq f_n < 2^n$ for all $n \geq 2$.

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## running time of euclid's algorithm

**Theorem:** Suppose that Euclid's algorithm takes $n$ steps for $\gcd(a, b)$ with $a > b$, then $a \geq f_{n+1}$

Set $r_{n+1} = a, r_n = b$ then Euclid's algorithm computes

$$
\begin{aligned}
r_{n+1} &= q_n r_n + r_{n-1} \\
r_n &= q_{n-1} r_{n-1} + r_{n-2} \\
&\vdots \\
r_3 &= q_2 r_2 + 1 \\
r_2 &= q_1 r_1
\end{aligned}
$$

each quotient $q_i \geq 1$
$r_1 \geq 1$