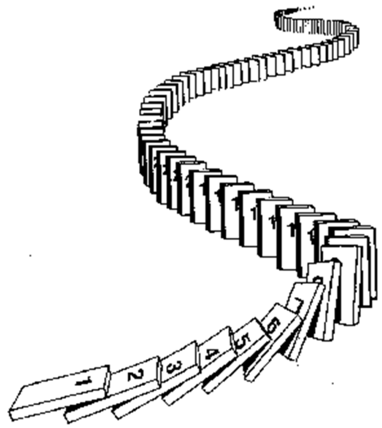
## CSE 311: Foundations of Computing

**Fall 2013**

**Lecture 14: Mathematical Induction**



## announcements

**Reading assignment**

**Induction**

5.1-5.2, 7th edition

4.1-4.2, 6th edition

## mathematical induction

**Method for proving statements about all integers $n \geq 0$.**

- **Part of sound logical inference that applies only in the domain of integers**
  - Not like scientific induction which is more like a guess from examples
- **Particularly useful for reasoning about programs since the statement might be "after n times through this loop, property P(n) holds"**

## finding a pattern

- $2^0 - 1 = 1 - 1 = 0 = 3 \cdot 0$
- $2^2 - 1 = 4 - 1 = 3 = 3 \cdot 1$
- $2^4 - 1 = 16 - 1 = 15 = 3 \cdot 5$
- $2^6 - 1 = 64 - 1 = 63 = 3 \cdot 21$
- $2^8 - 1 = 256 - 1 = 255 = 3 \cdot 85$
- ...

## how do you prove it?

Want to prove $3 \mid 2^{2n} - 1$ for all integers $n \geq 0$

- $n = 0$
- $n = 1$
- $n = 2$
- $n = 3$
- $\cdots$

## induction as a rule of Inference

Domain: integers ≥ 0

$$\frac{\begin{array}{l} P(0) \\ \forall\, k\, (P(k) \rightarrow P(k+1)) \end{array}}{\therefore\ \forall\, n\, P(n)}$$

## using the induction rule in a formal proof

$$\frac{\begin{array}{l} P(0) \\ \forall\, k\, (P(k) \rightarrow P(k+1)) \end{array}}{\therefore\ \forall\, n\, P(n)}$$

1. Prove P(0)
2. Let k be an arbitrary integer ≥ 0
   3. Assume that P(k) is true
   4. ...
   5. Prove P(k+1) is true
6. P(k) → P(k+1) — Direct Proof Rule
7. ∀ k (P(k) → P(k+1)) — Intro ∀ from 2-6
8. ∀ n P(n) — Induction Rule 1&7

## using the induction rule in a formal proof

$$\frac{\begin{array}{l} P(0) \\ \forall\, k\, (P(k) \rightarrow P(k+1)) \end{array}}{\therefore\ \forall\, n\, P(n)}$$

1. Prove P(0) — **Base Case**
2. Let k be an arbitrary integer ≥ 0
   3. Assume that P(k) is true — **Inductive Hypothesis** (2–3)
   4. ...
   5. Prove P(k+1) is true — **Inductive Step** (4–5)
6. P(k) → P(k+1) — Direct Proof Rule
7. ∀ k (P(k) → P(k+1)) — Intro ∀ from 2-6
8. ∀ n P(n) — Induction Rule 1&7

**Conclusion** (6–8)

## 5 steps to inductive proofs in english

**Proof:**

1. "By induction we will show that P(n) is true for every n≥0."
2. "Base Case:" Prove P(0)
3. "Inductive Hypothesis:"
   Assume P(k) is true for some arbitrary integer k ≥ 0"
4. "Inductive Step:" Want to prove that P(k+1) is true:
   Use the goal to figure out what you need.
   Make sure you are using I.H. and point out where you are using it. (Don't assume P(k+1) !!)
5. "Conclusion: Result follows by induction"

## induction example

Want to prove $3 \mid 2^{2n} - 1$ for all $n \geq 0$.

## $3 \mid 2^{2n} - 1$ for all $n \geq 0$.

## geometric sum

$1 + 2 + 4 + \cdots + 2^n = 2^{n+1} - 1$ for all $n \geq 0$

## $1 + 2 + 4 + \cdots + 2^n = 2^{n+1} - 1$ for all $n \geq 0$

## sum of first $n$ numbers

For all $n \geq 1$: $1 + 2 + \cdots + n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

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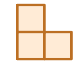
## harmonic numbers

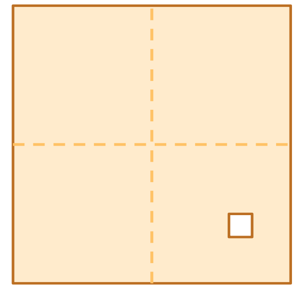
$$H_m = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{m} = \sum_{i=1}^{m} \frac{1}{i}$$

Prove that $H_{2^n} \geq 1 + \frac{n}{2}$ for all $n \geq 1$.

## Prove that $H_{2^n} \geq 1 + \frac{n}{2}$ for all $n \geq 1$.

## checkerboard tiling

Prove that a $2^n \times 2^n$ checkerboard with one square removed can be tiled with: 



## strong induction

$P(0)$

$\forall k \ \left( \left( P(0) \land P(1) \land P(2) \land \cdots \land P(k) \right) \rightarrow P(k+1) \right)$

$\therefore \forall n \ P(n)$

Follows from ordinary induction applied to
$Q(n) = P(0) \land P(1) \land P(2) \land \cdots \land P(n)$

## strong induction english proofs

1. By induction we will show that $P(n)$ is true for every $n \geq 0$
2. Base Case: Prove $P(0)$
3. Inductive Hypothesis:
   Assume that for some arbitrary integer $k \geq 0$, $P(j)$ is true for every $j$ from $0$ to $k$
4. Inductive Step:
   Prove that $P(k+1)$ is true using the Inductive Hypothesis (that $P(j)$ is true for all values $\leq k$)
5. Conclusion: Result follows by induction

## every integer $\geq 2$ is the product of primes

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## recursive definitions of functions

- $F(0) = 0$; $F(n + 1) = F(n) + 1$ for all $n \geq 0$
- $G(0) = 1$; $G(n + 1) = 2 \cdot G(n)$ for all $n \geq 0$
- $0! = 1$; $(n + 1)! = (n + 1) \cdot n!$ for all $n \geq 0$
- $H(0) = 1$; $H(n + 1) = 2^{H(n)}$ for all $n \geq 0$

## fibonacci numbers

$f_0 = 0$
$f_1 = 1$
$f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

## bounding the fibonacci numbers

**Theorem:** $2^{\frac{n}{2}-1} \leq f_n < 2^n$ **for all** $n \geq 2$.

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