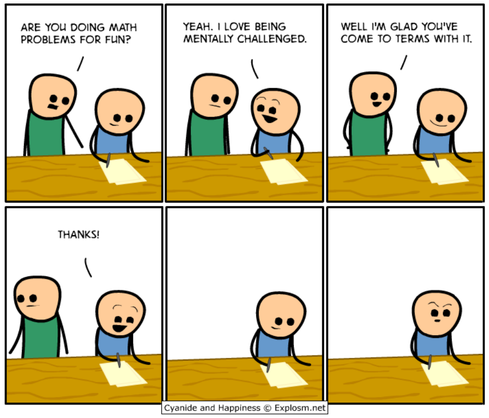
CSE 311: Foundations of Computing

Fall 2013

Lecture 13: Modular inverses, induction



announcements

Reading assignment

Induction

5.1-5.2, 7th edition

4.1-4.2, 6th edition

review: GCD

a = $2^3 \cdot 3 \cdot 5^2 \cdot 7 \cdot 11$ = 46,200

b = $2 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 13$ = 204,750

GCD(a, b) = $2^{\min(3,1)} \cdot 3^{\min(1,2)} \cdot 5^{\min(2,3)} \cdot 7^{\min(1,1)} \cdot 11^{\min(1,0)} \cdot 13^{\min(0,1)}$

Factoring is expensive!

Can we compute GCD(a,b) without factoring?

> If a and b are positive integers, then
> gcd(a,b) = gcd(b, a mod b)

review: euclid's algorithm

Repeatedly use the GCD fact to reduce numbers until you get $\mathrm{GCD}(x, 0) = x$.

GCD(660,126)= ?

660 = *5* • 126 + 30 GCD(660,126) = GCD(126,30)

126 = *4* • 30 + 6 = GCD(30,6)

30 = *5* • 6 + 0 = GCD(6,0)

= 6

bézout's theorem

If *a* and *b* are positive integers, then there exist integers ***s*** and ***t*** such that
gcd(*a*,*b*) = ***s****a* + ***t****b*.

extended euclid algorithm

- **Can use Euclid's Algorithm to find s, t such that** $\gcd(a, b) = sa + tb$
- e.g. gcd(35,27):

35 = *1* • 27 + 8	35 - *1* • 27 = 8
27= *3* • 8 + 3	27- *3* • 8 = 3
8 = *2* • 3 + 2	8 - *2* • 3 = 2
3 = *1* • 2 + 1	3 - *1* • 2 = 1
2 = *2* • 1 + 0	

- **Substitute back from the bottom**

1= 3 - *1* • 2 = 3 – *1* (8 - *2* • 3) = (-*1*) • 8 + *3* • 3
= (-*1*) • 8 + *3* (27- *3* • 8) = *3* • 27 + (-*10*) • 8
=

multiplicative inverse $\bmod m$

Suppose $\mathrm{GCD}(a, m) = 1$

By Bézout's Theorem, there exist integers s **and** t **such that** $sa + tm = 1$.

$s \bmod m$ **is the multiplicative inverse of** a:

$1 = (sa + tm) \bmod m = sa \bmod m$

solving modular equations

Solving $ax \equiv b \pmod{m}$ **for unknown** x **when** $\gcd(a, m) = 1$.

1. **Find** s **such that** $sa + tm = 1$
2. **Compute** $a^{-1} = s \bmod m$, **the multiplicative inverse of** a **modulo** m
3. **Set** $x = (a^{-1} \cdot b) \bmod m$

multiplicative cipher: $f(x) = ax \bmod m$

For a multiplicative cipher to be invertible:

$f(x) = ax\ mod\ m : \{0, \dots, m-1\} \to \{0, \dots, m-1\}$

must be one-to-one and onto

Lemma: If there is an integer b such that $ab \bmod m = 1$, then the function $f(x) = ax \bmod m$ is one-to-one and onto.

example

Solve: $7x \equiv 1 \pmod{26}$

mathematical induction

Method for proving statements about all integers $n \geq 0$.

- Part of sound logical inference that applies only in the domain of integers

  Not like scientific induction which is more like a guess from examples

- Particularly useful for reasoning about programs since the statement might be "after n times through this loop, property P(n) holds"

finding a pattern

- $2^0 - 1 = 1 - 1 = 0 = 3 \cdot 0$
- $2^2 - 1 = 4 - 1 = 3 = 3 \cdot 1$
- $2^4 - 1 = 16 - 1 = 15 = 3 \cdot 5$
- $2^6 - 1 = 64 - 1 = 63 = 3 \cdot 21$
- $2^8 - 1 = 256 - 1 = 255 = 3 \cdot 85$
- ...

how do you prove it?

Want to prove $3 \mid 2^{2n} - 1$ for all integers $n \geq 0$

- $n = 0$
- $n = 1$
- $n = 2$
- $n = 3$
- $\cdots$

induction as a rule of Inference

Domain: integers ≥ 0

$$\frac{\begin{array}{l} P(0) \\ \forall\, k\, (P(k) \ \rightarrow\ P(k+1)) \end{array}}{\therefore\ \forall\, n\, P(n)}$$

using the induction rule in a formal proof

$$\frac{\begin{array}{l} P(0) \\ \forall\, k\, (P(k) \ \rightarrow\ P(k+1)) \end{array}}{\therefore\ \forall\, n\, P(n)}$$

1. Prove P(0)
2. Let k be an arbitrary integer ≥ 0
    3. Assume that P(k) is true
    4. ...
    5. Prove P(k+1) is true
6. P(k) → P(k+1) — Direct Proof Rule
7. ∀ k (P(k) → P(k+1)) — Intro ∀ from 2-6
8. ∀ n P(n) — Induction Rule 1&7

using the induction rule in a formal proof

$$\frac{\begin{array}{l} P(0) \\ \forall\, k\, (P(k) \ \rightarrow\ P(k+1)) \end{array}}{\therefore\ \forall\, n\, P(n)}$$

1. Prove P(0) — **Base Case**
2. Let k be an arbitrary integer ≥ 0
    3. Assume that P(k) is true — **Inductive Hypothesis**
    4. ...
    5. Prove P(k+1) is true — **Inductive Step**
6. P(k) → P(k+1) — Direct Proof Rule
7. ∀ k (P(k) → P(k+1)) — Intro ∀ from 2-6
8. ∀ n P(n) — Induction Rule 1&7

Conclusion

5 steps to inductive proofs in english

Proof:

1. "By induction we will show that P(n) is true for every n≥0."
2. "Base Case:" Prove P(0)
3. "Inductive Hypothesis:"
   Assume P(k) is true for some arbitrary integer k ≥ 0"
4. "Inductive Step:" Want to prove that P(k+1) is true:
   Use the goal to figure out what you need.
   Make sure you are using I.H. and point out where you are using it. (Don't assume P(k+1) !!)
5. "Conclusion: Result follows by induction"

induction example

Want to prove $3 \mid 2^{2n} - 1$ for all $n \geq 0$.

$3 \mid 2^{2n} - 1$ for all $n \geq 0$.

geometric sum

$1 + 2 + 4 + \cdots + 2^n = 2^{n+1} - 1$ for all $n \geq 0$

$1 + 2 + 4 + \cdots + 2^n = 2^{n+1} - 1$ for all $n \geq 0$

sum of first n numbers

For all $n \geq 1$: $1 + 2 + \cdots + n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

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