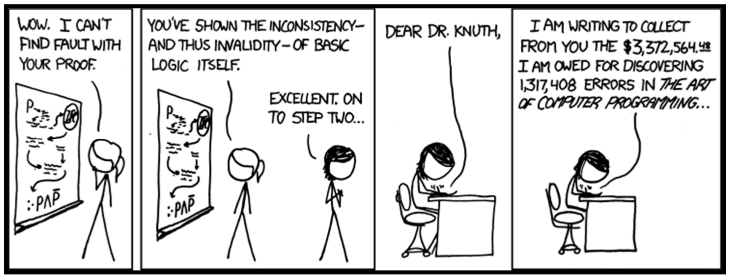
CSE 311: Foundations of Computing

Fall 2013

Lecture 6: Predicate logic, logical inference



announcements

Reading assignment

- **Logical inference**
  - 1.6-1.7 7th Edition
  - 1.5-1.7 6th Edition

review: predicate logic

predicate or **propositional function**

a function that returns a truth value, e.g.,

"x is a cat"

"x is prime"

"student x has taken course y"

"x > y"

"x + y = z" or Sum(x, y, z)

Predicates will have variables or constants as arguments.

review: quantifiers

$\forall$ x P(x)

P(x) is true for **every** x in the domain

read as "**for all x, P of x**"

$\exists$ x P(x)

There is an x in the domain for which P(x) is true

read as "**there exists x, (such that) P of x**"

review: statements with quantifiers

Domain:
Positive Integers

- $\exists$ x Even(x)
- $\forall$ x Odd(x)
- $\forall$ x (Even(x) $\vee$ Odd(x))
- $\exists$ x (Even(x) $\wedge$ Odd(x))
- $\forall$ x Greater(x+1, x)
- $\exists$ x (Even(x) $\wedge$ Prime(x))

review: statements with quantifiers

Domain:
Positive Integers

- $\forall$ x $\exists$ y Greater (y, x)
- $\forall$ x $\exists$ y Greater (x, y)
- $\forall$ x $\exists$ y (Greater(y, x) $\wedge$ Prime(y))
- $\forall$ x (Prime(x) $\rightarrow$ (Equal(x, 2) $\vee$ Odd(x))
- $\exists$ x $\exists$ y (Sum(x, 2, y) $\wedge$ Prime(x) $\wedge$ Prime(y))

review: statements with quantifiers

Domain:
Positive Integers

Even(x)
Odd(x)
Prime(x)
Greater(x,y)
Equal(x,y)

- **"There is an odd prime"**
- **"If x is greater than two, x is not an even prime"**
- $\forall$x$\forall$y$\forall$z ((Sum(x, y, z) $\wedge$ Odd(x) $\wedge$ Odd(y))$\rightarrow$ Even(z))
- **"There exists an odd integer that is the sum of two primes"**

review: English to predicate logic

"Red cats like tofu"

Cat(x)
Red(x)
LikesTofu(x)

goldbach's conjecture

"Every even integer greater than two can be expressed as the sum of two primes"

Even(x)
Odd(x)
Prime(x)
Greater(x,y)
Sum(x,y,z)

Domain:
Positive Integers

scope of quantifiers

example: Notlargest(x) $\equiv \exists$ y Greater (y, x)
$\equiv \exists$ z Greater (z, x)

truth value:

doesn't depend on y or z "bound variables"

does depend on x "free variable"

quantifiers only act on free variables of the formula they quantify

$\forall x (\exists y (P(x,y) \rightarrow \forall x\, Q(y, x)))$

scope of quantifiers

$\exists x\ (P(x) \wedge Q(x))$ vs. $\exists x\, P(x) \wedge \exists x\, Q(x)$

nested quantifiers

- bound variable names don't matter

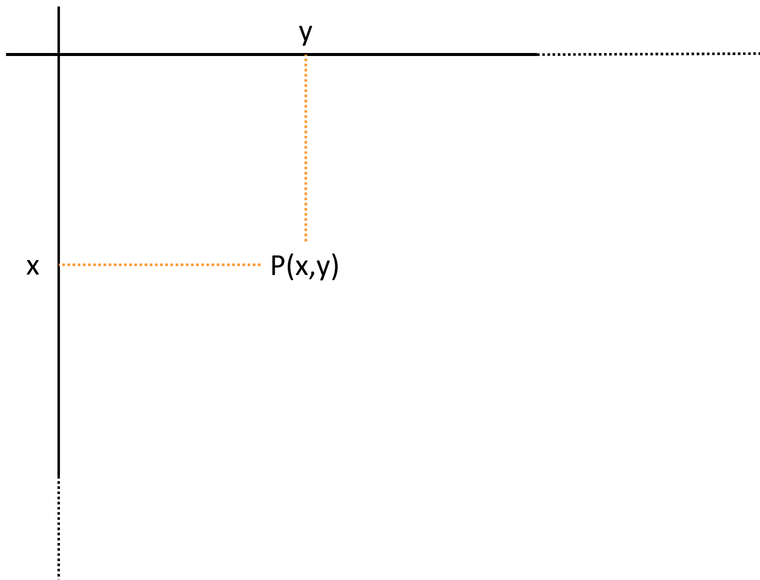
  $\forall x \exists y\, P(x, y) \equiv \forall a \exists b\, P(a, b)$

- positions of quantifiers can sometimes change

  $\forall x\, (Q(x) \wedge \exists y\, P(x, y)) \equiv \forall x \exists y\, (Q(x) \wedge P(x, y))$

- but: order is important...

predicate with two variables



quantification with two variables

expression	when **true**	when **false**
$\forall x \forall y\, P(x, y)$		
$\exists x \exists y\, P(x, y)$		
$\forall x \exists y\, P(x, y)$		
$\exists y \forall x\, P(x, y)$		

negations of quantifiers

- not every positive integer is prime
- some positive integer is not prime
- prime numbers do not exist
- every positive integer is not prime

de morgan's laws for quantifiers

$$\neg \forall x\ P(x) \equiv \exists x \neg P(x)$$
$$\neg \exists x\ P(x) \equiv \forall x \neg P(x)$$

de morgan's laws for quantifiers

$$\neg \forall x\ P(x) \equiv \exists x \neg P(x)$$
$$\neg \exists x\ P(x) \equiv \forall x \neg P(x)$$

"There is no largest integer"

$$\neg \exists x \forall y\ (x \geq y)$$
$$\equiv \forall x \neg \forall y\ (x \geq y)$$
$$\equiv \forall x\ \exists y \neg (x \geq y)$$
$$\equiv \forall x\ \exists y\ \ (y > x)$$

"For every integer there is a larger integer"

logical Inference

- So far we've considered:
  - How to understand and *express* things using propositional and predicate logic
  - How to *compute* using Boolean (propositional) logic
  - How to show that different ways of expressing or computing them are *equivalent* to each other
- Logic also has methods that let us *infer* implied properties from ones that we know
  - Equivalence is a small part of this

applications of logical inference

- Software Engineering
  - Express desired properties of program as set of logical constraints
  - Use inference rules to show that program implies that those constraints are satisfied
- Artificial Intelligence
  - Automated reasoning
- Algorithm design and analysis
  - e.g., Correctness, Loop invariants.
- Logic Programming, e.g. Prolog
  - Express desired outcome as set of constraints
  - Automatically apply logic inference to derive solution

proofs

- Start with hypotheses and facts
- Use rules of inference to extend set of facts
- Result is proved when it is included in the set

an inference rule: *Modus Ponens*

- If p and p $\rightarrow$ q are both true then q must be true
- Write this rule as $\frac{p,\ p \rightarrow q}{\therefore\ q}$
- Given:
  - If it is Monday then you have a 311 class today.
  - It is Monday.
- Therefore, by modus ponens:
  - You have a 311 class today.

proofs

Show that r follows from p, p $\rightarrow$ q, and q $\rightarrow$ r

1.	p	given
2.	$p \rightarrow q$	given
3.	$q \rightarrow r$	given
4.	q	modus ponens from 1 and 2
5.	r	modus ponens from 3 and 4

proofs can use equivalences too

Show that $\neg p$ follows from $p \rightarrow q$ and $\neg q$

1.	$p \rightarrow q$	given
2.	$\neg q$	given
3.	$\neg q \rightarrow \neg p$	contrapositive of 1
4.	$\neg p$	modus ponens from 2 and 3

inference rules

- Each inference rule is written as: $\frac{A,\ B}{\therefore\ C, D}$
  ...which means that if both A and B are true then you can infer C and you can infer D.
  - For rule to be correct $(A \wedge B) \rightarrow C$ and $(A \wedge B) \rightarrow D$ must be a tautologies
- Sometimes rules don't need anything to start with. These rules are called axioms:
  - e.g. *Excluded Middle Axiom* $\frac{}{\therefore\ p \vee \neg p}$

simple propositional inference rules

Excluded middle plus two inference rules per binary connective, one to eliminate it and one to introduce it

$$\frac{p \wedge q}{\therefore p, q} \qquad \frac{p, q}{\therefore p \wedge q}$$

$$\frac{p \vee q, \neg p}{\therefore q} \qquad \frac{p}{\therefore p \vee q, q \vee p}$$

$$\frac{p, p \rightarrow q}{\therefore q} \qquad \frac{p \Rightarrow q}{\therefore p \rightarrow q}$$

Direct Proof Rule

Not like other rules

important: applications of inference rules

- You can use equivalences to make substitutions of any sub-formula.
- Inference rules only can be applied to whole formulas (not correct otherwise).

e.g. 1. $p \rightarrow q$ given

~~2. $(p \vee r) \rightarrow q$ intro $\vee$ from 1.~~

Does not follow! e.g . p=F, q=F, r=T

direct proof of an implication

- **$p \Rightarrow q$** denotes a proof of q given p as an assumption
- **The direct proof rule:**

  If you have such a proof then you can conclude that $p \rightarrow q$ is true

Example:

proof subroutine

1.	p	assumption
2.	$p \vee q$	intro for $\vee$ from 1
3.	$p \rightarrow (p \vee q)$	direct proof rule