CSE311: Final Review Answers

1. Break the equivalence into two implications:

$$\neg(p \leftrightarrow q) \equiv \neg((p \rightarrow q) \wedge (p \leftarrow q)) \equiv$$

$$\neg((\neg p \vee q) \wedge (p \vee \neg q)) \equiv (\neg(\neg p \vee q)) \vee (\neg(p \vee \neg q)) \equiv$$

$$(p \wedge \neg q) \vee (\neg p \wedge q)$$

2. Answers:

   (a) False. Take $q = T, p = F$.

   (b) True. Break the implications and by some refactoring you get a tautology.

   (c) True.

3. Showed in class. We take the all assignments that have one or three of w, x, y, z set to 1 and take the sum of products in the standard way.

4. First direction:

$$A \subseteq B$$

$$\forall x \ (x \in A \rightarrow x \in B)$$

$$\forall x \ (x \notin B \rightarrow x \notin A)$$

$$\overline{B} \subseteq \overline{A}$$

   All I used is the definition of a subset and the contrapositive of an implication. The other direction can be solved in the exact same way.

5. Did this in the quiz section. The first two are pretty standard. For the third, take its negation, simplify it with De Morgan's law and stick an inverter at the end of the circuit. The final form is $\overline{x + y + z}$.

6. You can construct a truth table and take the sum of products. Then turn it into a circuit. The resulting circuit is going to be fairly large. Another approach would be to take every possible combination of three variables and feed it into an AND gate (fan-in is three here but you can simulate it with two AND gates). Then take the OR of the resulting outputs. This works since if we have a majority of ones, at least a subset of the variables of size three will be set to one. If such a subset does not exist, it means that at least three variables are set to zero.

7. Answers:

   (a) Not reflexive because of $(1, 1)$ missing. Not symmetric because of $(4, 2)$ missing. Not antisymmetric because we have $(2, 3), (3, 2)$ but 2 and 3 are not the same. Transitive.

(b) Reflexive. Symmetric. Not antisymmetric. Transitive.

(c) Obviously not reflexive. Symmetric. Not antisymmetric since 2 and 4 are different elements. Not transitive, since we also need $(2,2),(4,4)$.

8. Answers:

   (a) $\forall x\ (P(x) \rightarrow H(x))$

   (b) $\exists x\ \neg H(x) \wedge P(x)$

9. This is a way to look at this: The states capture the remainder of the division of the difference in ones and zeroes by three. We have one state for 0, one for 1 and one for 2. The transitions should capture the fact that when we get a zero, this remainder should grow. When we get a one, the opposite happens.

10. Basis:

$$0 \in S$$

$$5 \in S$$

    Recurrence:

$$x \in S \rightarrow (x+2) \in S$$

11. The union is reflexive but not necessarily transitive. Take a look at:

    `http://www.proofwiki.org/wiki/Union_of_Transitive_Relations_Not_Always_Transitive`

12. Answers:

    (a) $V_{n+1} = 2V_n + 1$

    (b) Basis is $V_0 = 1 \leq 3^0 = 1$. Our hypothesis is that for $n = k \geq 0, V_k \leq 3^k$. Now we have to prove that the statement holds for $n = k+1$:

$$V_k \leq 3^k \Leftrightarrow 2V_k + 1 \leq 2 * 3^k + 1 \Leftrightarrow$$

$$V_{k+1} \leq 2 * 3^k + 1 \leq 2 * 3^k + 3^k = 3 * 3^k = 3^{k+1}$$