CSE311: Worksheet, May 17, 2012

1. Take a look at the url indicated in the worksheet

2. The problem is in the inductive step. Notice that if I choose n to be equal to 2, then the inductive step says that $gcd(f(3), f(2))$ reduces to $gcd(f(2), f(1))$ in one step. Notice that $f(3) = 2$ and $f(2) = 1$. By applying one step of Euclid's algorithm on $gcd(2, 1)$ we get $gcd(1, 0) = gcd(f(0), f(1))$ and not $gcd(f(2), f(1))$.

3. We will apply strong induction. Consider the single vertex tree. It satisfies the statement, since it has one leaf and no internal nodes. Now assume that the statement holds for all trees with up to k leaves (hypothesis). We will prove that it also holds for trees with $k + 1$ leaves. Let T be such a tree. Then it must be the result of the application of the recurrence on two smaller trees T_1, T_2. Let l_1, l_2 denote the number of leaves of T_1, T_2. We have that $l_1 \geq 1, l_2 \geq 1$ and $l_1 + l_2 = k + 1$. From this we get that $l_1, l_2 \leq k$. Therefore we can apply the hypothesis and deduce the fact that the two trees have $l_1 - 1$ and $l_2 - 1$ internal nodes respectively. By conjoining T_1, T_2 via a common root we add just one internal node. The total number of internal nodes is $l_1 - 1 + l_2 - 1 + 1 = (l_1 + l_2) - 1 = (k + 1) - 1$. Therefore the statement holds.

4. The problem here is the constant term at the rhs of the equation. If we try to apply standard induction techniques to approach this, we will soon find ourselves in a dead-end (I invite you to try it). We will solve this by actually proving a stronger statement:

$$\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - \frac{1}{n}$$

   If we prove this, then the original statement follows, since $2 - \frac{1}{n} < 2$. Our basis is $n = 1$ for which we have that $\frac{1}{1} \leq 2 - \frac{1}{1}$, which holds with equality. Now assume that the statement holds for $n = k$ (hypothesis):

$$\sum_{i=1}^{k} \frac{1}{i^2} \leq 2 - \frac{1}{k}$$

   We will prove that:

$$\sum_{i=1}^{k+1} \frac{1}{i^2} \leq 2 - \frac{1}{k+1}$$

   Notice that:

$$\sum_{i=1}^{k+1} \frac{1}{i^2} = \sum_{i=1}^{k} \frac{1}{i^2} + \frac{1}{(k+1)^2}$$

By the hypothesis:

$$\sum_{i=1}^{k} \frac{1}{i^2} + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2}$$

Now all that is left is to prove that:

$$2 - \frac{1}{k} + \frac{1}{(k+1)^2} < 2 - \frac{1}{k+1}$$

which by transitivity of inequality, concludes the proof.

$$2 - \frac{1}{k} + \frac{1}{(k+1)^2} < 2 - \frac{1}{k+1} \Leftrightarrow \frac{1}{k+1} < \frac{1}{k} - \frac{1}{(k+1)^2}, k \geq 1 \Leftrightarrow$$

$$\frac{1}{(k+1)^2} < \frac{1}{k} - \frac{1}{k+1} \Leftrightarrow \frac{1}{(k+1)^2} < \frac{1}{k(k+1)}$$

which holds.