CSE 311 Foundations of Computing I

Lecture 11
Modular Arithmetic
Autumn 2012

Announcements

- Reading assignments
  - Today and Friday:
    - 4.1-4.3 7th Edition
    - 3.5, 3.6 6th Edition
    - 2.5, 2.6 up to p. 191 5th Edition
- Homework 4
  - Available now

Highlights from last lecture: Set Theory

$x \in A$: "x is an element of A"
$x \notin A$: $\neg (x \in A)$

$A = B \equiv \forall x\, (x \in A \leftrightarrow x \in B)$

$(A \subseteq B \wedge B \subseteq A) \rightarrow A = B$

$A \cup B = \{ x \mid (x \in A) \vee (x \in B) \}$

Applications of Set Theory

- Implementation: Characteristic Vector
- Private Key Cryptography
- Unix File Permissions

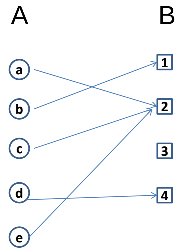
Russell's Paradox

$$S = \{ x \mid x \notin x \}$$

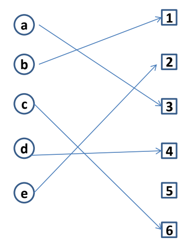
Functions review

- A *function* from *A* to *B*
  - an assignment of exactly one element of *B* to each element of *A*.
  - We write *f: A→B.*
  - "Image of *a" = f(a)*
- *Domain* of *f* : A
- *Range* of *f* = set of all images of elements of A

Image, Preimage



Is this a function? one-to-one? onto?



Number Theory (and applications to computing)

- Branch of Mathematics with direct relevance to computing
- Many significant applications
  - Cryptography
  - Hashing
  - Security
- Important tool set

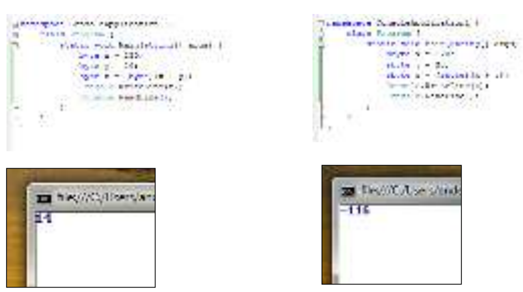
Modular Arithmetic

- Arithmetic over a finite domain
- In computing, almost all computations are over a finite domain

What are the values computed?

```
public void Test1() {
        byte x = 250;
        byte y = 20;
        byte z = (byte) (x + y);
        Console.WriteLine(z);
}
```

```
public void Test2() {
        sbyte x = 120;
        sbyte y = 20;
        sbyte z = (sbyte) (x + y);
        Console.WriteLine(z);
}
```



Arithmetic mod 7

- a $+_7$ b = (a + b) mod 7
- a $\times_7$ b = (a $\times$ b) mod 7

+	0	1	2	3	4	5	6
0							
1							
2							
3							
4							
5							
6							

x	0	1	2	3	4	5	6
0							
1							
2							
3							
4							
5							
6							

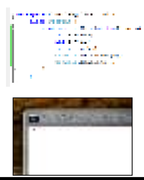
Divisibility

Integers a, b, with a ≠ 0, we say that a *divides* b is there is an integer k such that b = ak. The notation a | b denotes a divides b.

Division Theorem

Let *a* be an integer and *d* a positive integer. Then there are *unique* integers *q* and *r*, with 0 ≤ *r* < *d*, such that *a* = *dq* + *r*.

q = *a* **div** *d* *r* = *a* **mod** *d*



Note: r ≥ 0 even if a < 0. Not quite the same as `a%d`

Modular Arithmetic

Let a and b be integers, and m be a positive integer. We say a *is congruent to b modulo m* if m divides a – b. We use the notation a ≡ b (mod m) to indicate that a is congruent to b modulo m.

Modular arithmetic

Let a and b be integers, and let m be a positive integer. Then a ≡ b (mod m) if and only if a mod m = b mod m.

Modular arithmetic

Let m be a positive integer. If a ≡ b (mod m) and c ≡ d (mod m), then
- a + c ≡ b + d (mod m) and
- ac ≡ bd (mod m)

Example

Let n be an integer, prove that $n^2 \equiv 0 \pmod 4$ or $n^2 \equiv 1 \pmod 4$

n-bit Unsigned Integer Representation

- Represent integer x as sum of powers of 2:
  If $x = \sum_{i=0}^{n-1} b_i 2^i$ where each $b_i \in \{0,1\}$
  then representation is $b_{n-1} \ldots b_2 \, b_1 \, b_0$

  99 = 64 + 32 + 2 + 1
  18 = 16 + 2

- For n = 8:
  99: 0110 0011
  18: 0001 0010

Signed integer representation

n-bit signed integers
Suppose $-2^{n-1} < x < 2^{n-1}$
First bit as the sign, n-1 bits for the value

99 = 64 + 32 + 2 + 1
18 = 16 + 2

For n = 8:
99: 0110 0011
-18: 1001 0010

Any problems with this representation?

Two's complement representation

n bit signed integers, first bit will still be the sign bit
Suppose $0 \le x < 2^{n-1}$, x is represented by the binary representation of x
Suppose $0 < x \le 2^{n-1}$, -x is represented by the binary representation of 2^n-x

Key property: Two's complement representation of any number y is equivalent to y mod 2^n so arithmetic works mod 2^n

99 = 64 + 32 + 2 + 1
18 = 16 + 2

For n = 8:
99: 0110 0011
-18: 1110 1110

Signed vs Two's complement

-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
1111	1110	1101	1100	1011	1010	1001	0000	0001	0010	0011	0100	0101	0110	0111

Signed

-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
1000	1001	1010	1011	1100	1101	1110	1111	0000	0001	0010	0011	0100	0101	0110	0111

Two's complement

Two's complement representation

- Suppose $0 < x \le 2^{n-1}$, -x is represented by the binary representation of 2^n-x
- To compute this: Flip the bits of x then add 1:
  - All 1's string is 2^n-1 so
    - Flip the bits of x $\equiv$ replace x by 2^n-1-x