CSE 311 Foundations of Computing I

Lecture 7
Logical Inference
Autumn 2012

Announcements

- Reading assignments
  - Logical Inference
    - 1.6, 1.7 7th Edition
    - 1.5, 1.6, 1.7 6th Edition
    - 1.5, 3.1 5th Edition

Highlights from last lecture

- Predicates
  - Cat(x), Prime(x), HasTaken(s,c)
- Quantifiers
  - $\forall x$ (Even(x) $\vee$ Odd(x)), $\exists x$ (Cat(x) $\wedge$ LikesTofu(x))
- Order of quantifiers
  - $\forall x \exists y$ Greater (y, x), $\exists y \forall x$ Greater (y, x)
- Correspondence between world and logic
  - "Red cats like tofu"
  - $\forall$ x ((Cat(x) $\wedge$ Red(x)) $\rightarrow$ LikesTofu(x))

Scope of Quantifiers

- Notlargest(x) $\equiv \exists y$ Greater (y, x)
  $\equiv \exists z$ Greater (z, x)
  - Value doesn't depend on y or z "bound variables"
  - Value does depend on x "free variable"

- Quantifiers only act on free variables of the formula they quantify
  - $\forall$ x ($\exists$ y (P(x,y) $\rightarrow \forall$ x Q(y, x)))

Scope of Quantifiers

- $\exists x$ (P(x) $\wedge$ Q(x)) vs $\exists x$ P(x) $\wedge$ $\exists x$ Q(x)

Nested Quantifiers

- Bound variable name doesn't matter
  - $\forall$ x $\exists$ y P(x, y) $\equiv \forall$ a $\exists$ b P(a, b)

- Positions of quantifiers can change
  - $\forall$ x (Q(x) $\wedge \exists$ y P(x, y)) $\equiv \forall$ x $\exists$ y (Q(x) $\wedge$ P(x, y))

- BUT: Order is important...

Quantification with two variables

Expression	When true	When false
$\forall x \forall y P(x, y)$		
$\exists x \exists y P(x, y)$		
$\forall x \exists y P(x, y)$		
$\exists y \forall x P(x, y)$		

Negations of Quantifiers

- Not every positive integer is prime
- Some positive integer is not prime
- Prime numbers do not exist
- Every positive integer is not prime

De Morgan's Laws for Quantifiers

$$\neg\forall x\ P(x) \equiv \exists x\ \neg P(x)$$
$$\neg\exists x\ P(x) \equiv \forall x\ \neg P(x)$$

De Morgan's Laws for Quantifiers

$$\neg\forall x\ P(x) \equiv \exists x\ \neg P(x)$$
$$\neg\exists x\ P(x) \equiv \forall x\ \neg P(x)$$

"There is no largest integer"

$$\neg \exists x \forall y\ (x \geq y)$$
$$\equiv \forall x \neg \forall y\ (x \geq y)$$
$$\equiv \forall x\ \exists y \neg (x \geq y)$$
$$\equiv \forall x\ \exists y\ (y > x)$$

"For every integer there is a larger integer"

Logical Inference

- So far we've considered
  - How to understand and *express* things using propositional and predicate logic
  - How to *compute* using Boolean (propositional) logic
  - How to show that different ways of expressing or computing them are *equivalent* to each other
- Logic also has methods that let us *infer* implied properties from ones that we know
  - Equivalence is a small part of this

Applications of Logical Inference

- Software Engineering
  - Express desired properties of program as set of logical constraints
  - Use inference rules to show that program implies that those constraints are satisfied
- AI
  - Automated reasoning
- Algorithm design and analysis
  - e.g., Correctness, Loop invariants.
- Logic Programming, e.g. Prolog
  - Express desired outcome as set of constraints
  - Automatically apply logic inference to derive solution

Proofs

- Start with hypotheses and facts
- Use rules of inference to extend set of facts
- Result is proved when it is included in the set

An inference rule: *Modus Ponens*

- If p and $p \rightarrow q$ are both true then q must be true
- Write this rule as $\dfrac{p,\ p \rightarrow q}{\therefore\ q}$
- Given:
  - If it is Monday then you have a 311 class today.
  - It is Monday.
- Therefore, by Modus Ponens:
  - You have a 311 class today

Proofs

- Show that r follows from p , $p \rightarrow q$, and $q \rightarrow r$

1. p Given
2. $p \rightarrow q$ Given
3. $q \rightarrow r$ Given
4. q Modus Ponens from 1 and 2
5. r Modus Ponens from 3 and 4

Inference Rules

- Each *inference rule* is written as $\dfrac{A,\ B}{\therefore\ C, D}$ which means that if both A and B are true then you can infer C and you can infer D.
  - For rule to be correct $(A \wedge B) \rightarrow C$ and $(A \wedge B) \rightarrow D$ must be a tautologies
- Sometimes rules don't need anything to start with. These rules are called *axioms*:
  - e.g. *Excluded Middle Axiom* $\dfrac{\quad}{\therefore\ p \vee \neg p}$

Simple Propositional Inference Rules

- Excluded middle plus two inference rules per binary connective, one to eliminate it and one to introduce it

$$\frac{p \wedge q}{\therefore\ p,\ q} \qquad \frac{p,\ q}{\therefore\ p \wedge q}$$

$$\frac{p \vee q\ ,\ \neg p}{\therefore\ q} \qquad \frac{p}{\therefore\ p \vee q,\ q \vee p}$$

$$\frac{p,\ p \rightarrow q}{\therefore\ q} \qquad \frac{p \Rightarrow q}{\therefore\ p \rightarrow q}$$

Direct Proof Rule

Not like other rules! See next slide…

Direct Proof of an Implication

- $p \Rightarrow q$ denotes a proof of q given p as an assumption. Don't confuse with $p \rightarrow q$.
- The direct proof rule
  - if you have such a proof then you can conclude that $p \rightarrow q$ is true
- E.g.

Proof subroutine for $p \Rightarrow (p \vee q)$

1. p Assumption
2. $p \vee q$ Intro for $\vee$ from 1

3. $p \rightarrow (p \vee q)$ Direct proof rule

Proofs can use Equivalences too

Show that ¬p follows from p→q and ¬q

1. p→q Given
2. ¬q Given
3. ¬q → ¬p Contrapositive of 1
4. ¬p Modus Ponens from 2 and 3

Inference Rules for Quantifiers

P(c) for some c
∴ ∃ x P(x)

∀ x P(x)
∴ P(a) for any a

"Let a be anything"...P(a)
∴ ∀ x P(x)

∃ x P(x)
∴ P(c) for some special c

Proofs using Quantifiers

- Show that "Simba is a cat" follows from "All lions are cats" and "Simba is a lion" (using the domain of all animals)

Proofs using Quantifiers

- "There exists an even prime number"

General Proof Strategy

A. Look at the rules for introducing connectives to see how you would build up the formula you want to prove from pieces of what is given
B. Use the rules for eliminating connectives to break down the given formulas so that you get the pieces you need to do A.
C. Write the proof beginning with B followed by A.