1. "Define the Fibonnaci numbers as follows: $f(0) = 0, f(1) = 1$, and $f(n) = f(n-2) + f(n-1)$ for all integers $n > 1$. Prove by induction that, for all nonnegative integers n, the number of iterations used by Euclid's algorithm to compute $gcd(f(n+1), f(n))$ is n."

   Proof: The basis is $n = 0$, and indeed $gcd(1, 0)$ uses no iterations. For the induction step, the first iteration changes the arguments from $(f(n+1), f(n))$ to $(f(n), f(n-1))$, and the induction hypothesis says it takes $n-1$ more iterations to finish the computation.

   The only hitch is that the theorem is false for almost all values of n. For your entertainment, find the flaw in the proof. (It's not hard to find once you know it's false, but I find the proof absolutely convincing if you don't suspect it's false.)

   Answer:
   The problem is in the inductive step. Notice that if I choose n to be equal to 2, then the inductive step says that $gcd(f(3), f(2))$ reduces to $gcd(f(2), f(1))$ in one step. Notice that $f(3) = 2$ and $f(2) = 1$. By applying one step of Euclids algorithm on $gcd(2, 1)$ we get $gcd(1, 0) = gcd(f(0), f(1))$ and not $gcd(f(2), f(1))$.

2. Prove the following:

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + ... + \frac{1}{n^2} \leq 2 \ , \ n \geq 1$$

   Hint1: Try replacing the right hand side of the inequality with something that will make the statement stronger.

   Answer:
   The problem here is the constant term at the rhs of the equation. If we try to apply standard induction techniques to approach this, we will soon find ourselves in a dead-end (I invite you to try it). We will solve this by actually proving a stronger statement:

$$\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - \frac{1}{n}$$

   If we prove this, then the original statement follows, since $2 - \frac{1}{n} < 2$. Our basis is $n = 1$ for which we have that $\frac{1}{1} \leq 2 - \frac{1}{1}$, which holds with equality. Now assume that the statement holds for $n = k$ (hypothesis):

$$\sum_{i=1}^{k} \frac{1}{i^2} \leq 2 - \frac{1}{k}$$

We will prove that:

$$\sum_{i=1}^{k+1} \frac{1}{i^2} \leq 2 - \frac{1}{k+1}$$

Notice that:

$$\sum_{i=1}^{k+1} \frac{1}{i^2} = \sum_{i=1}^{k} \frac{1}{i^2} + \frac{1}{(k+1)^2}$$

By the hypothesis:

$$\sum_{i=1}^{k} \frac{1}{i^2} + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2}$$

Now all that is left is to prove that:

$$2 - \frac{1}{k} + \frac{1}{(k+1)^2} < 2 - \frac{1}{k+1}$$

which by transitivity of inequality, concludes the proof.

$$2 - \frac{1}{k} + \frac{1}{(k+1)^2} < 2 - \frac{1}{k+1} \Leftrightarrow \frac{1}{k+1} < \frac{1}{k} - \frac{1}{(k+1)^2}, k \geq 1 \Leftrightarrow$$

$$\frac{1}{(k+1)^2} < \frac{1}{k} - \frac{1}{k+1} \Leftrightarrow \frac{1}{(k+1)^2} < \frac{1}{k(k+1)}$$

which holds.