CSE 311 Quiz Section: December 6, 2012 (Solutions)

1 Determining Countability

Determine whether each of these sets is finite, countably infinite, or uncountable. For those that are countably infinite, exhibit a one-to-one correspondence between the set of positive integers and that set.

a) the integers that are multiples of 7
Answer: Countably infinite; listing: 0, 7, -7, 14, -14, 21, -21, 28, -28, ...

b) the integers less than 100
Answer: Countably infinite; listing: 99, 98, 97, ... , 0, -1, -2, -3, ...

c) the real numbers between 0 and $\frac{1}{2}$
Answer: Uncountable (similar to diagonalization proof for real numbers between 0 and 1)

d) the real numbers not containing 0 in their decimal representations

Answer: Uncountable

e) all bit strings not containing the bit 0
Answer: Countably infinite; listing: $\lambda, 1, 11, 111, 1111, ...$

f) all positive rational numbers that cannot be written with denominators less than 4
Answer: Countably infinite; use 'dovetailing' argument similar to positive rationals except omit fractions with denominators less than four and those that reduce to fractions with a denominator less than four (as well as repeats)

2 Sets and Countability

a) Show that if A and B are sets, A is uncountable, and $A \subseteq B$, then B is uncountable.

Answer: Assume B is countable. Then the elements of B can be listed $b_1, b_2, b_3, \ldots$ Because A is a subset of B, taking the subsequence of $\{b_n\}$ that contains the terms that are in A gives a listing of elements of A. But we assumed A is uncountable, therefore we have reached a contradiction. Hence B is uncountable.

b) If A is an uncountable set and B is a countable set, must $A - B$ be uncountable?

Answer: Assume $A - B$ is countable. Then, since $A = (A - B) \cup (A \cap B)$, and $A \cap B$ is countably infinite because B is countable, the elements of A can be listed in a sequence by alternating elements of $A - B$ and the elements of $A \cap B$ (because they are both countably infinite, then a listing exists for each $A - B$ and $A \cap B$). However, finding a listing means that A is countable, but we assumed A was uncountable. Therefore, yes, $A - B$ must be uncountable.