Name: ______________________________________
Student ID: ______________________________________

SOLUTIONS

CSE 311 Winter 2011: Midterm Exam

(closed book, closed notes except for 1-page summary)
Total: 100 points, 5 questions. Time: 50 minutes

Instructions:

1. Write your name and student ID on the first sheet (once you start, write your last name on all sheets). Write or mark your answers in the space provided. If you need more space or scratch paper, additional sheets will be available. Make sure you write down the question number and your name on any additional sheets.
2. Tables for logical equivalence and set identities are included in the back.
3. Read all questions carefully before answering them. Feel free to come to the front to ask for clarifications.
4. *Hint 1*: You may answer the questions in any order, so if you find that you're having trouble with one of them, move on to another one that seems easier.
5. *Hint 2*: If you don't know the answer to a question, don't omit it - do the best you can! You may still get partial credit for whatever you wrote down. Good luck!

Do not start until you are given the "green signal"…

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1. **(20 points: 4 points each) Circle True (T) or False (F) below. Very briefly justify your answers** (e.g. by giving an example or counter-example, by citing a theorem or result we proved in class, or by *briefly* sketching a proof or construction).

   a. For all sets A and B, $A \times B = B \times A$...T F
   Why/Why not?
   FALSE. Counterexample: Let A = {a}, B = {1}. Then, $A \times B = \{(a,1)\}$ and $B \times A = \{(1,a)\}$. Since $(a,1) \neq (1,a)$, $A \times B \neq B \times A$.

   b. $\forall n \in Z\ \exists m \in Z\ (nm = m)$ ....T F
   Why/Why not?
   TRUE. For any $n \in Z$, choose m = 0. Then, nm = 0 = m.

   c. For all integers m and n, mn is even if and only if m and n are both even.......T F
   Why/Why not?
   FALSE. Counterexample: Let m = 3 and n = 2. Then, mn = 6 is even but m is not even.

   d. The function $f(x,y) = x^y$ from $Z^+ \times Z^+$ to Z^+ is onto...T F
   Why/Why not?
   TRUE. Proof: Let z be any element of Z^+. Choose x = z and y = 1. Then, $f(x,y) = f(z,1) = z^1 = z$. Thus, $\forall z \in Z^+\ \exists (x,y) \in Z^+ \times Z^+$ s.t. f(x,y) = z. Therefore, f is onto.

   e. The number 1 is among the numbers generated by the linear congruential pseudorandom number generator: $x_{n+1} = 2x_n$ **mod** 9 with seed $x_0 = 2$............T F
   Why/Why not?
   TRUE. The generator generates the numbers 2, 4, 8, 7, 5, 1, 2, 4, …
   Therefore, 1 is among the numbers generated.

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2. (20 points: 10 points each) Propositional Logic

a. Show that $\neg(p \land (p \to q)) \equiv (p \to \neg q)$ using known logical equivalences (see tables at back of this exam).

$\neg(p \land (p \to q))$	**$\equiv \neg p \lor \neg(p \to q)$**	**De Morgan's law**
	$\equiv \neg p \lor (p \land \neg q)$	**Logical Equivalence Table 7**
	$\equiv (\neg p \lor p) \land (\neg p \lor \neg q)$	**Distributive law**
	$\equiv (p \lor \neg p) \land (\neg p \lor \neg q)$	**Commutative law**
	$\equiv T \land (\neg p \lor \neg q)$	**Negation law**
	$\equiv (\neg p \lor \neg q) \land T$	**Commutative law**
	$\equiv (\neg p \lor \neg q)$	**Identity law**
	$\equiv (p \to \neg q)$	**Logical Equivalence Table 7**

b. Use a truth table to show that $[\neg p \land (p \lor q)] \to q$ is a tautology.

p	**q**	**$\neg p$**	**$p \lor q$**	**$\neg p \land (p \lor q)$**	**$[\neg p \land (p \lor q)] \to q$**
T	**T**	**F**	**T**	**F**	**T**
T	**F**	**F**	**T**	**F**	**T**
F	**T**	**T**	**T**	**T**	**T**
F	**F**	**T**	**F**	**F**	**T**

Because $[\neg p \land (p \lor q)] \to q$ is always true, it is a tautology.

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3. (18 points: 10 and 8 points) Predicate Logic

a. Let S(x,y) be the predicate "x has seen the movie y" and R(x,y,z) the predicate "x has recommended movie y to z," where the domain of x and z is the set of all students in this class and the domain of y is the set of all movies. Use only the quantifiers ∃ and ∀ to express the following statements about students in the class:

   i. There is a movie that everyone has seen and no one has recommended.

   $\exists y \forall x (S(x,y) \land \forall z \neg R(x,y,z))$

   ii. There is someone in the class who has not seen any movies but to whom at least two students have recommended a movie.

   $\exists x \forall y (\neg S(x,y) \land \exists x_1 \exists x_2 \exists y_1 \exists y_2 ((x_1 \neq x_2) \land R(x_1, y_1,x) \land R(x_2, y_2,x))$

b. Express the negations of each of these statements so that all negation symbols immediately precede predicates:

   i. $\exists x(\neg P(x) \land \exists y \forall z (\neg Q(x,y,z) \rightarrow R(x,y,z)))$

   $\forall x(P(x) \lor \forall y \exists z (\neg Q(x,y,z) \land \neg R(x,y,z)))$

   ii. $\forall x ([\exists y \exists z ((x = z) \lor R(x,y,z))] \leftrightarrow [\forall y \forall z (Q(x,y,z) \land (y \neq z))])$

   (Use Table 8: negate $p \leftrightarrow q$ as $p \leftrightarrow \neg q$)

   $\exists x ([\exists y \exists z ((x = z) \lor R(x,y,z))] \leftrightarrow [\exists y \exists z (\neg Q(x,y,z) \lor (y = z))])$

   <u>OR</u>

   (Negate $p \leftrightarrow q$ as $\neg p \leftrightarrow q$)

   $\exists x ([\forall y \forall z ((x \neq z) \land \neg R(x,y,z))] \leftrightarrow [\forall y \forall z (Q(x,y,z) \land (y \neq z))])$

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4. (22 points: 12 and 10 points) Rules of Inference and Proofs

a. Use rules of inference and logical equivalences to show that the hypotheses $\neg p$, $q \wedge s$, $(r \wedge s) \rightarrow p$, and $q \rightarrow (r \vee t)$ imply the conclusion t.

1.	$\neg p$	**Hypothesis**
2.	$(r \wedge s) \rightarrow p$	**Hypothesis**
3.	$\neg(r \wedge s)$	**Modus Tollens using 1 and 2**
4.	$\neg r \vee \neg s$	**De Morgan's law**
5.	$q \wedge s$	**Hypothesis**
6.	q	**Simplification of 5**
7.	$q \rightarrow (r \vee t)$	**Hypothesis**
8.	$(r \vee t)$	**Modus Ponens using 6 and 7**
9.	$t \vee \neg s$	**Resolution using 8 and 4**
10.	$\neg s \vee t$	**Commutative law**
11.	s	**Simplification of 5**
12.	t	**Disjunctive syllogism using 10 and 11 (or equivalently, simplified resolution using 10 and 11)**

b. Suppose there are 14 stores in a mall. Prove that at least 3 in a group of 30 people shopping at stores in the mall must be shopping at the same store.

Proof by contradiction: Assume at most 2 people in a group of 30 people are shopping at the same store. Since there are 14 stores, at most 28 people can be shopping at all stores in the mall. This contradicts the fact that there are 30 people shopping. Therefore, at least 3 people must be shopping at the same store.

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5. (20 points: 10 points each) Sets and Number Theory

a. Let A, B, C be three sets. Use set builder notation and logical equivalences to show that $A - (B \cap C) = (A - B) \cup (A - C)$.

$\mathbf{A - (B \cap C) = \{x \mid x \in A \wedge x \notin (B \cap C)\}}$	**Definition of set difference**
$\mathbf{= \{x \mid x \in A \wedge \neg(x \in (B \cap C))\}}$	**Definition of** $\notin$
$\mathbf{= \{x \mid x \in A \wedge \neg(x \in B \wedge x \in C)\}}$	**Definition of** $\cap$
$\mathbf{= \{x \mid x \in A \wedge (x \notin B \vee x \notin C)\}}$	**De Morgan's law**
$\mathbf{= \{x \mid (x \in A \wedge x \notin B) \vee (x \in A \wedge x \notin C)\}}$	**Distributive law**
$\mathbf{= \{x \mid (x \in (A-B)) \vee (x \in (A-C))\}}$	**Definition of set difference**
$\mathbf{= \{x \mid x \in (A-B) \cup (A-C)\}}$	**Definition of union**
$\mathbf{= (A-B) \cup (A-C)}$	**Set builder notation**

b. State whether the following statements are TRUE or FALSE (no proofs needed):

i. $\lceil 3.5 \rceil + \lfloor -3.5 \rfloor = 1$
   FALSE. ($\lceil 3.5 \rceil + \lfloor -3.5 \rfloor = 4 + (-4) = 0$)

ii. For any integer x and any positive integer n, if $n \mid x$, then $n \mid (x + 3n)$
   TRUE. ($n \mid x$ and $n \mid 3n$, therefore $n \mid (x + 3n)$ – theorem in class/test)

iii. $311 \equiv 299 \pmod 6$
   TRUE. (because $6 \mid (311-299)$)

iv. gcd(16,18) = 2.
   TRUE. ($16 = 2^4$ and $18 = 2 \cdot 3^2$, greatest common divisor is 2)

v. The word "AXE" is encrypted as "DAH" by Caesar's cipher.
   TRUE. (Shift each letter by 3 mod 26)

END OF EXAM

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TABLE 6 Logical Equivalences.

Equivalence	*Name*
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

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TABLE 7 Logical Equivalences Involving Conditional Statements.
$p \rightarrow q \equiv \neg p \vee q$
$p \rightarrow q \equiv \neg q \rightarrow \neg p$
$p \vee q \equiv \neg p \rightarrow q$
$p \wedge q \equiv \neg(p \rightarrow \neg q)$
$\neg(p \rightarrow q) \equiv p \wedge \neg q$
$(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$
$(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$
$(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$
$(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$

TABLE 8 Logical Equivalences Involving Biconditionals.
$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$
$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

TABLE 2 Rules of Inference for Quantified Statements.

Rule of Inference	*Name*
$\forall x P(x)$ $\therefore P(c)$	Universal instantiation
$P(c)$ for an arbitrary c $\therefore \forall x P(x)$	Universal generalization
$\exists x P(x)$ $\therefore P(c)$ for some element c	Existential instantiation
$P(c)$ for some element c $\therefore \exists x P(x)$	Existential generalization

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TABLE 1 Rules of Inference.

Rule of Inference	*Tautology*	*Name*
p $p \to q$ $\therefore q$	$[p \wedge (p \to q)] \to q$	Modus ponens
$\neg q$ $p \to q$ $\therefore \neg p$	$[\neg q \wedge (p \to q)] \to \neg p$	Modus tollens
$p \to q$ $q \to r$ $\therefore p \to r$	$[(p \to q) \wedge (q \to r)] \to (p \to r)$	Hypothetical syllogism
$p \vee q$ $\neg p$ $\therefore q$	$[(p \vee q) \wedge \neg p] \to q$	Disjunctive syllogism
p $\therefore p \vee q$	$p \to (p \vee q)$	Addition
$p \wedge q$ $\therefore p$	$(p \wedge q) \to p$	Simplification
p q $\therefore p \wedge q$	$[(p) \wedge (q)] \to (p \wedge q)$	Conjunction
$p \vee q$ $\neg p \vee r$ $\therefore q \vee r$	$[(p \vee q) \wedge (\neg p \vee r)] \to (q \vee r)$	Resolution

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TABLE 1 Set Identities.

Identity	*Name*
$A \cup \emptyset = A$ $A \cap U = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive laws
$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws