Name Student ID

CSE 311: Foundations of Computing I
Spring 2011
Final exam

Instructions

1. Start at 2:30PM and stop at 4:20PM.
2. There are 7 questions, worth a total of 150 points.
3. Write your name and student ID on the first sheet (once you start, write your last name on all sheets). Write or mark your answers in the space provided. If you need more space or scratch paper, additional sheets will be available. Make sure you write down the question number and your name on any additional sheets.
4. You may bring in two double-sided sheets of notes. Apart from this, you may not use books, notes or calculators.
5. *Suggestion:* Skim all questions before starting, and answer the questions that seem easiest and fastest to answer first.
6. *Suggestion:* If you don't know the answer to a question, try to clearly write down what you know anyway, and you may still be able to get partial credit.

1. **Logic, proofs, sets and functions.** *(25 points; 5+10+10)*

   (a) Prove or disprove: $\exists x \in \mathbb{R}^+, \forall y \in \mathbb{R}(y \geq x \to y^2 \geq 2y)$.

   (b) Let $P(S)$ denote the power set of S; i.e. $P(S) = \{T : T \subseteq S\}$. Prove that $A \subseteq B$ if and only if $P(A) \subseteq P(B)$.

   (c) Let S and T be subsets of a universal set U, and define $A_{0,0} = S \cap T$, $A_{0,1} = S \cap \bar{T}$, $A_{1,0} = \bar{S} \cap T$ and $A_{1,1} = \bar{S} \cap \bar{T}$. Express $S \cup T$ as a union of some or all of the $\{A_{0,0}, A_{0,1}, A_{1,0}, A_{1,1}\}$. You do not need to prove your answer. *Hint: You may find a Venn diagram helpful, although it is not required.*

   (a) Choose $x = 2$. Then we use a direct proof to show that $y \geq x \to y^2 \geq 2y$. Assume that $y \geq x = 2$. Since $y \geq 0$, we can multiply both sides by y and still have a valid inequality: $y^2 \geq 2y$. QED

   (b) For one direction, assume that $A \subseteq B$. We will use a direct proof to show that $\forall S\,(S \in P(A) \to S \in P(B))$.

$S \in P(A)$	by assumption
$S \subseteq A$	by the definition of a power set
$S \subseteq B$	using the fact that $A \subseteq B$
$S \in P(B)$	by the definition of a power set

   Since $\forall S\,(S \in P(A) \to S \in P(B))$, we have that $P(A) \subseteq P(B)$.
   For the other direction, assume that $P(A) \subseteq P(B)$.

$P(A) \subseteq P(B)$	by assumption	(1)
$A \subseteq A$	set identity (this step could be skipped)	(2)
$A \in P(A)$	definition of power set	(3)
$A \in P(B)$	by (1) and (3)	(4)
$A \subseteq B$	definition of power set	(5)

   (c) $S \cup T = A_{0,0} \cup A_{0,1} \cup A_{1,0}$.

2. **Number theory.** *(25 points; 5+10+10)*

   (a) Use Euclid's algorithm to compute the gcd of 328 and 432. Write down the numbers you obtain at the intermediate steps.

   (b) Prove that if $a, b \in \mathbb{Z}$ and $b > 0$, then there exist unique $q, r \in \mathbb{Z}$ satisfying $a = bq - r$ (note the $-$ here) and $0 \leq r < b$.

   (c) One type of cicada living in the Eastern US has a lifecycle of 17 years, has appeared in 1970, 1987, 2004, and next will appear in 2021. Suppose that a parasite that attacks the cicadas has an n-year lifecycle, and also appeared in 1970, then $1970 + n$, $1970 + 2n$, etc. Assume that $1 \leq n \leq 16$. If the cicadas and parasites both appeared in the same year in 1970, in what year will they next both appear?

3. **Induction and recursion.** *(30 points; 10+20)*

   (a) Prove using induction that $\sum_{k=1}^{n} k^2 = n(n+1)(2n+1)/6$ for all positive integers n.

   (b) Euclid's algorithm for computing the GCD of a pair of positive integers a, b is as follows:

EUCLID(a, b):
If $(a < b)$ return EUCLID(b, a)
If $b = 0$ return a
Use the division algorithm to compute $q, r \in \mathbb{Z}$ such that $a = bq + r$ and $0 \leq r < b$.
Return EUCLID(b, r)

Define $P(a)$ to the predicate that EUCLID(a, b) returns $\gcd(a, b)$ for all $0 \leq b < a$. Use strong induction to prove that EUCLID(a, b) =gcd(a, b) for all positive integers a, b.

4. **Relations.** *(15 points; 5+10)*
   (a) Define the rock-paper-scissors relation on $S = \{r, p, s\}$ by $R = \{(r, r), (p, p), (s, s), (p, r), (r, s), (s, p)\}$. Is this relation a partial order? Why or why not?
   (b) Consider the relation R on $\mathbb{R}$ given by $\{(x, y) | x - y \in \mathbb{Z}\}$.
      i. Prove that R is an equivalence relation.
      ii. What is the equivalence class of 1? What is the equivalence class of 0.5?

5. **Graphs and trees.** *(15 points; 5+10)*
   (a) Define the complete graph K_n to be the undirected graph on n vertices with no self-loops and with all possible edges present. Prove by induction that K_n has $\sum_{k=1}^{n-1} k$ edges.
   (b) Draw a directed graph with four vertices such that the edges form a partial order. Your score on this question will be 1 point per edge that you draw, or 0 if what you draw isn't a partial order.

6. **Circuits and boolean algebra.** *(15 points)* The goal of this problem is to prove that AND and OR are not functionally complete. Let $x_1, \ldots, x_n$ be boolean variables for some $n \geq 1$. We say that a boolean function $F(x_1, \ldots, x_n)$ is monotone if

$$\forall x_1, \ldots, x_n \in \{0, 1\}, \forall i \in [n] \, (F(x_1, \ldots, x_n) = 1 \to F(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n) = 1).$$

   In other words, if F equals 1 for some input, then changing one of those inputs to 1 will not change F.
   (a) Suppose that $F(x_1, \ldots, x_n)$ is a boolean function constructed from AND and OR gates. Prove, using structural induction, that F is monotone.
   (b) Give an example of a boolean function that is not monotone.

7. **Turing Machines and Finite state machines.** *(25 points)*
   (a) Draw a DFA that accepts the same strings as the NFA in Figure 1.
   (b) Construct a Turing machine that takes as input a binary string, and halts in an accepting state with the entire tape filled with blank symbols and with the tape head in its starting position.

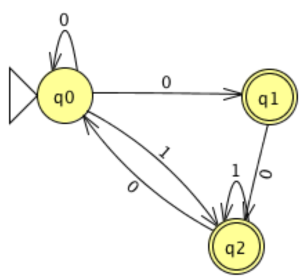

Figure 1: A non-deterministic finite automaton.