CSE 311 Foundations of Computing I

Lecture 26

Cardinality, Countablity & Computability

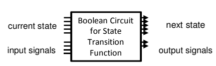
Autumn 2011

Announcements

- Reading
  - 7th edition: 2.5 (Cardinality) + p. 201 and 13.5
  - 6th edition: pp. 158-160 (Cardinality)+ p 177 and 12.5
  - 5th edition: Pages 233-236 (Cardinality), p. ? and 11.5
- Homework 10 out today, due next Friday
  - Homework 9 due today

Last lecture highlights

- Sequential Circuits for FSMs
  - Combinational logic for transition function



  - Sequential logic for iteration
- Carry-look-ahead Adders
  - $C_4=G_4+G_3P_4+G_2P_3P_4+G_1P_2P_3P_4+G_0P_1P_2P_3P_4$ etc.
- Composition trees and Parallel Prefix

Computing & Mathematics

Computers as we know them grew out of a desire to avoid bugs in mathematical reasoning

A Brief History of Reasoning

- Ancient Greece
  - Deductive logic
    - Euclid's Elements
  - Infinite things are a problem
    - Zeno's paradox

A Brief History of Reasoning

- 1670's-1800's Calculus & infinite series
  - Suddenly infinite stuff really matters
  - Reasoning about infinite still a problem
    - Tendency for buggy or hazy proofs
- Mid-late 1800's
  - Formal mathematical logic
    - Boole Boolean Algebra
  - Theory of infinite sets and cardinality
    - Cantor

      "There are more real #'s than rational #'s"

A Brief History of Reasoning

- 1900
  - Hilbert's famous speech outlines goal: mechanize all of mathematics
    - 23 problems
- 1930's
  - Gödel, Turing show that Hilbert's program is impossible.
    - Gödel's Incompleteness Theorem
    - Undecidability of the Halting Problem

Both use ideas from Cantor's proof about reals & rationals

A Brief History of Reasoning

- 1930's
  - How can we formalize what algorithms are possible?
    - Turing machines (Turing, Post)
      - basis of modern computers
    - Lambda Calculus (Church)
      - basis for functional programming
    - μ-recursive functions (Kleene)
      - alternative functional programming basis

All are equivalent!

Turing Machines

Church-Turing Thesis

Any reasonable model of computation that includes all possible algorithms is equivalent in power to a Turing machine

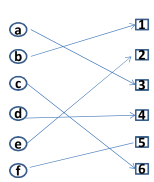
- Evidence
  - Huge numbers of equivalent models to TM's based on radically different ideas

Starting with Cantor

- How big is a set?
  - If S is finite, we already defined |S| to be the number of elements in S.
  - What if S is infinite? Are all of these sets the same size?
    - Natural numbers $\mathbb{N}$
    - Even natural numbers
    - Integers $\mathbb{Z}$
    - Rational numbers $\mathbb{Q}$
    - Real numbers $\mathbb{R}$

Cardinality

Def: Two sets A and B are the same size (same cardinality) iff there is a 1-1 and onto function f:A→B



Also applies to infinite sets

Cardinality

- The natural numbers and even natural numbers have the same cardinality:

0 1 2 3 4 5 6 7 8 9 10 ...

0 2 4 6 8 10 12 14 16 18 20 ...

n is matched with 2n

Countability

Definition: A set is *countable* iff it is the same size as some subset of the natural numbers

Equivalent: A set S is *countable* iff there is an onto function g: $\mathbb{N} \rightarrow S$

Equivalent: A set S is *countable* iff we can write $S=\{s_1, s_2, s_3, \ldots\}$

The set of all integers is countable

Is the set of positive rational numbers countable?

- We can't do the same thing we did for the integers
  - Between any two rational numbers there are an infinite number of others

Positive Rational Numbers

1/1	**1/2**	**1/3**	**1/4**	**1/5**	**1/6**	**1/7**	**1/8**	**…**
2/1	**2/2**	**2/3**	**2/4**	**2/5**	**2/6**	**2/7**	**2/8**	**…**
3/1	**3/2**	**3/3**	**3/4**	**3/5**	**3/6**	**3/7**	**3/8**	**…**
4/1	**4/2**	**4/3**	**4/4**	**4/5**	**4/6**	**4/7**	**4/8**	**…**
5/1	**5/2**	**5/3**	**5/4**	**5/5**	**5/6**	**5/7**	**…**	
6/1	**6/2**	**6/3**	**6/4**	**6/5**	**6/6**	**…**		
7/1	**7/2**	**7/3**	**7/4**	**7/5**	**….**			
…	**…**	**…**	**…**	**…**				

{Positive Rational Numbers} is Countable

1/1	**1/2**	**1/3**	**1/4**	**1/5**	**1/6**	**1/7**	**1/8**	**…**
2/1	**2/2**	**2/3**	**2/4**	**2/5**	**2/6**	**2/7**	**2/8**	**…**
3/1	**3/2**	**3/3**	**3/4**	**3/5**	**3/6**	**3/7**	**3/8**	**…**
4/1	**4/2**	**4/3**	**4/4**	**4/5**	**4/6**	**4/7**	**4/8**	**…**
5/1	**5/2**	**5/3**	**5/4**	**5/5**	**5/6**	**5/7**	**…**	
6/1	**6/2**	**6/3**	**6/4**	**6/5**	**6/6**	**…**		
7/1	**7/2**	**7/3**	**7/4**	**7/5**	**….**			
…	**…**	**…**	**…**	**…**				

{Positive Rational Numbers} is Countable

$\mathbb{Q}^+$ = **{1/1, 2/1,1/2, 3/1,2/2,1/3, 4/1,2/3,3/2,1/4, 5/1,4/2,3/3,2/4,1/5, …}**

List elements in order of

- numerator+denominator
- breaking ties according to denominator
  - Only k numbers when the total is k

Technique is called "dovetailing"

Claim: Σ* is countable for every finite Σ

The set of all Java programs is countable

What about the Real Numbers?

Q: Is *every* set is countable?

A: Theorem [Cantor] The set of real numbers (even just between 0 and 1) is NOT countable

Proof is by contradiction using a new method called "diagonalization"

Proof by contradiction

- Suppose that $\mathbb{R}^{[0,1)}$ is countable
- Then there is some listing of all elements
  $\mathbb{R}^{[0,1)} = \{ r_1, r_2, r_3, r_4, \ldots \}$
- We will prove that in such a listing there must be at least one missing element which contradicts statement "$\mathbb{R}^{[0,1)}$ is countable"
- The missing element will be found by looking at the decimal expansions of $r_1, r_2, r_3, r_4, \ldots$

Real numbers between 0 and 1: $\mathbb{R}^{[0,1)}$

- Every number between 0 and 1 has an infinite decimal expansion:

  1/2 = 0.50000000000000000000000...

  1/3 = 0.33333333333333333333333...

  1/7 = 0.14285714285714285714285...

  π -3 = 0.14159265358979323846264...

  1/5 = 0.19999999999999999999999...

  = 0.20000000000000000000000...

Representations as decimals

Representation is unique except for the cases that decimal ends in all 0's or all 9's.

x = 0.19999999999999999999999...

10x = 1.9999999999999999999999...

9x = 1.8 so x=0.200000000000000000...

Won't allow the representations ending in all 9's

All other representations give elements of $\mathbb{R}^{[0,1)}$

Supposed Listing of $\mathbb{R}^{[0,1)}$

		1	2	3	4	5	6	7	8	9	...
r_1	0.	5	0	0	0	0	0	0	0	...	...
r_2	0.	3	3	3	3	3	3	3	3	...	...
r_3	0.	1	4	2	8	5	7	1	4	...	...
r_4	0.	1	4	1	5	9	2	6	5	...	...
r_5	0.	1	2	1	2	2	1	2	2	...	...
r_6	0.	2	5	0	0	0	0	0	0	...	...
r_7	0.	7	1	8	2	8	1	8	2	...	...
r_8	0.	6	1	8	0	3	3	9	4	...	...
...	....	...	....	....	...	...	...	...	...	...	

Supposed Listing of $\mathbb{R}^{[0,1)}$

		1	2	3	4	5	6	7	8	9	...
r_1	0.	5	0	0	0	0	0	0	0	...	...
r_2	0.	3	3	3	3	3	3	3	3	...	...
r_3	0.	1	4	2	8	5	7	1	4	...	...
r_4	0.	1	4	1	5	9	2	6	5	...	...
r_5	0.	1	2	1	2	2	1	2	2	...	...
r_6	0.	2	5	0	0	0	0	0	0	...	...
r_7	0.	7	1	8	2	8	1	8	2	...	...
r_8	0.	6	1	8	0	3	3	9	4	...	...
...	....	...	....	....	....	...	...	...	...	...	

Flipped Diagonal

Flipping Rule:

If digit is 5, make it 1
If digit is not 5, make it 5

		1	2	3	4	5	6	7	8	9	...
r_1	0.	5 1	0	0	0						
r_2	0.	3	3 5	3	3						
r_3	0.	1	4	2 5	8	5	7	1	4	...	...
r_4	0.	1	4	1	5 1	9	2	6	5	...	...
r_5	0.	1	2	1	2	2 5	1	2	2	...	...
r_6	0.	2	5	0	0	0	0 5	0	0	...	...
r_7	0.	7	1	8	2	8	1	8 5	2	...	...
r_8	0.	6	1	8	0	3	3	9	4 5	...	...
...	....	...	....	....	...	...	...	...	...	... ...	

Flipped Diagonal Number D

		1	2	3	4	5	6	7	8	9	...
D =	0.	1	5	5	1	5	5	5	5	...	

D is in $\mathbb{R}^{[0,1)}$

But for all **n**, we have $D \neq r_n$ since they differ on n^{th} digit (which is not **0** or **9**)

⇒ list was incomplete

⇒ $\mathbb{R}^{[0,1)}$ is not countable

The set of all functions f : ℕ→{0,1,...,9} is not countable

Non-computable functions

- We have seen that
  - The set of all (Java) programs is countable
  - The set of all functions f : ℕ→{0,1,...,9} is not countable
- So...
  - There must be some function f : ℕ→{0,1,...,9} that is not computable by any program!