CSE 311 Foundations of Computing I

Lecture 15

Recursive Definitions and Structural Induction

Autumn 2011

Announcements

- Reading assignments
  - Today:
    - 5.3 7th Edition
    - 4.3 6th Edition
    - 3.4 5th Edition (not all there)
- Midterm Friday, Nov 4
  - Closed book, closed notes
  - Sample midterm questions available on website
  - Extra office hours Thursday, times TBA

Highlights from last lecture

- Strong Induction

  P(0)

  $\underline{\forall k\ ((P(0) \wedge P(1) \wedge P(2) \wedge \ldots \wedge P(k)) \rightarrow P(k+1))}$

  $\therefore \forall n\ P(n)$
- Strong Induction proof layout:
  1. By induction we will show that P(n) is true for every n≥0
  2. Base Case: Prove P(0)
  3. Inductive Hypothesis: Assume that for some arbitrary integer k ≥ 0, P(j) is true for every j from 0 to k
  4. Inductive Step: Prove that P(k+1) is true using Inductive Hypothesis that P(j) is true for all values ≤ k
  5. Conclusion: Result follows by induction

Fibonacci Numbers

- $f_0 = 0$; $f_1 = 1$; $f_n = f_{n-1} + f_{n-2}$

Bounding the Fibonacci Numbers

- Theorem: $2^{n/2-1} \leq f_n < 2^n$ for $n \geq 2$

Fibonacci numbers and the running time of Euclid's algorithm

- Theorem: Suppose that Euclid's algorithm takes n steps for gcd(*a*,*b*) with *a*>*b*, then $a \geq f_{n+1}$ so $a \geq 2^{(n-1)/2}$
  - # of steps at most twice # of bits of *a*
- Set $r_{n+1}=a$, $r_n=b$ then Euclid's alg. computes

$r_{n+1}= q_n r_n + r_{n-1}$

$r_n= q_{n-1} r_{n-1} + r_{n-2}$

•••

$r_3= q_2 r_2 + r_1$

$r_2= q_1 r_1$

each quotient $q_i \geq 1$

$r_1 \geq 1$

Recursive Definitions of Sets

- Recursive definition
  - Basis step: $0 \in S$
  - Recursive step: if $x \in S$, then $x + 2 \in S$
  - Exclusion rule: Every element in S follows from basis steps and a finite number of recursive steps

Recursive definitions of sets

Basis: $6 \in S$; $15 \in S$;
Recursive: if $x, y \in S$, then $x + y \in S$;

Basis: $[1, 1, 0] \in S$, $[0, 1, 1] \in S$;
Recursive:
if $[x, y, z] \in S$, α in R, then $[\alpha x, \alpha y, \alpha z] \in S$
if $[x_1, y_1, z_1]$, $[x_2, y_2, z_2] \in S$
then $[x_1 + x_2, y_1 + y_2, z_1 + z_2] \in S$

Powers of 3

Recursive Definitions of Sets: General Form

- Recursive definition
  - *Basis step:* Some specific elements are in S
  - *Recursive step:* Given some existing named elements in S some new objects constructed from these named elements are also in S.
  - Exclusion rule: Every element in S follows from basis steps and a finite number of recursive steps

Structural Induction: proving properties of recursively defined sets

How to prove $\forall x \in S.\ P(x)$ is true:

•**Base Case:** Show that P is true for all specific elements of S mentioned in the *Basis step*

•**Inductive Hypothesis:** Assume that P is true for some arbitrary values of each of the existing named elements mentioned in the *Recursive step*

•**Inductive Step:** Prove that P holds for each of the new elements constructed in the *Recursive step* using the named elements mentioned in the Inductive Hypothesis

•Conclude that $\forall x \in S.\ P(x)$

Structural Induction versus Ordinary Induction

- Ordinary induction is a special case of structural induction:
  - Recursive Definition of $\mathbb{N}$
    - Basis: $0 \in \mathbb{N}$
    - Recursive Step: If $k \in \mathbb{N}$ then $k+1 \in \mathbb{N}$
- Structural induction follows from ordinary induction
  - Let Q(n) be true iff for all $x \in S$ that take n Recursive steps to be constructed, P(x) is true.

Using Structural Induction

- Let S be given by
  - Basis: $6 \in S$; $15 \in S$;
  - Recursive: if $x, y \in S$, then $x + y \in S$.
- Claim: Every element of S is divisible by 3

Strings

- An *alphabet* Σ is any finite set of characters.
- The set Σ* of *strings* over the alphabet Σ is defined by
  - Basis: λ ∈ S (λ is the empty string)
  - Recursive: if w ∈ Σ*, x ∈ Σ, then wx ∈ Σ*

Function definitions on recursively defined sets

Len(λ) = 0;
Len(wx) = 1 + Len(w); for w ∈ Σ*, x ∈ Σ

Concat(w, λ) = w for w ∈ Σ*
Concat(w_1,w_2x) = Concat(w_1,w_2)x for w_1, w_2 in Σ*, x ∈ Σ