CSE 311 Foundations of Computing I

Lecture 13

Mathematical Induction

Autumn 2011

Announcements

- Reading assignments
  - Today:
    - 5.1 7th Edition
    - 4.1 6th Edition
    - 3.2 5th Edition
- Homework 5
  - Coming soon . . .

Highlights from last lecture

- Greatest common divisor (gcd)
  - Definition and computation via prime factorization
  - Euclid's algorithm 78=2 x 33 + 12
    33=2 x 12 + 9
    12=1 x 9 + 3
    9=3 x 3 so gcd(78,33)=3
  - Bezout: ∃ s,t such that gcd(a,m)=sa+tm
    - E.g. 3=1x12-1x9 =1x12-1x(33-2x12) = -1x33+3x12
      = -1x33+3x(78-2x33)= 3x78-7x33
  - Uniquely solve $ax \equiv b \pmod{m}$ for $0 \le x < m$ if $\gcd(a,m)=1$

Induction Example

- Want to prove $3 \mid 2^{2n} - 1$ for all $n \ge 0$
  - n=0
  - n=1
  - n=2
  - n=3
  - ...

Induction as a rule of Inference

Domain: integers ≥ 0

$$\begin{array}{l} P(0) \\ \underline{\forall k\,(P(k) \rightarrow P(k+1))} \\ \therefore \forall n\, P(n) \end{array}$$

How would we use the induction rule in a formal proof?

$$\begin{array}{l} P(0) \\ \underline{\forall k\,(P(k) \rightarrow P(k+1))} \\ \therefore \forall n\, P(n) \end{array}$$

1. Prove P(0)
2. Let k be an arbitrary integer ≥ 0
   3. Assume that P(k) is true
   4. ...
   5. Prove P(k+1) is true
6. P(k) → P(k+1) Direct Proof Rule
7. ∀ k (P(k) → P(k+1)) Intro ∀ from 2-6
8. ∀ n P(n) Induction Rule 1&7

How would we use the induction rule in a formal proof?

P(0)
$\forall$ k (P(k) $\rightarrow$ P(k+1))
$\therefore$ $\forall$ n P(n)

1. Prove P(0) — **Base Case**
2. Let k be an arbitrary integer ≥ 0 — **Inductive Hypothesis**
   3. Assume that P(k) is true
      4. ... — **Inductive Step**
      5. Prove P(k+1) is true
6. P(k) $\rightarrow$ P(k+1) — Direct Proof Rule
7. $\forall$ k (P(k) $\rightarrow$ P(k+1)) — Intro $\forall$ from 2-6
8. $\forall$ n P(n) — Induction Rule 1&7

Conclusion

5 Steps to Inductive Proofs in English

Proof:
1. "By induction we will show that P(n) is true for every n≥0"
2. "Base Case:" Prove P(0)
3. "Inductive Hypothesis: Assume that P(k) is true for some arbitrary integer k ≥ 0"
4. "Inductive Step:" Want to prove that P(k+1) is true:
   Use the goal to figure out what you need.
   Make sure you are using I.H. and point out where you are using it. (Don't assume P(k+1)!)
5. "Conclusion: Result follows by induction"

Induction Example

- Want to prove $3 \mid 2^{2n} - 1$ for all $n \geq 0$

$1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$ for all $n \geq 0$

$1+2+\ldots+n = \sum_{i=1}^{n} i = n(n+1)/2$ for all $n \geq 1$

Harmonic Numbers

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \frac{1}{n} = \sum_{k=1}^{n} \frac{1}{k}$$

Prove $H_{2^n} \geq 1 + \frac{n}{2}$ for all $n \geq 1$

Cute Application: Checkerboard Tiling with Trinominos

Prove that a $2^n \times 2^n$ checkerboard with one square removed can be tiled with: