CSE 311 Foundations of Computing I

Lecture 5, Boolean Logic and Predicates

Autumn 2011

Announcements

- Reading assignments
  - Boolean Algebra
    - 12.1 – 12.3 7th Edition
    - 11.1 – 11.3 6th Edition
    - 10.1 – 10.3 5th Edition
  - Predicates and Quantifiers
    - 1.4, 1.5 7th Edition
    - 1.3, 1.4 5th and 6th Edition

Highlights from last lecture

- Boolean algebra to circuit design
- Boolean algebra
  - a set of elements B = {0, 1}
  - binary operations { + , • }
  - and a unary operation { ’ }
  - such that the following axioms hold:

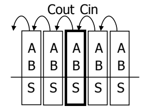
George Boole – 1854

1. the set B contains at least two elements: a, b

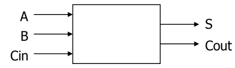
2. closure:	a + b is in B	a • b is in B
3. commutativity:	a + b = b + a	a • b = b • a
4. associativity:	a + (b + c) = (a + b) + c	a • (b • c) = (a • b) • c
5. identity:	a + 0 = a	a • 1 = a
6. distributivity:	a + (b • c) = (a + b) • (a + c)	a • (b + c) = (a • b) + (a • c)
7. complementarity:	a + a’ = 1	a • a’ = 0

A simple example: 1-bit binary adder

- Inputs: A, B, Carry-in
- Outputs: Sum, Carry-out



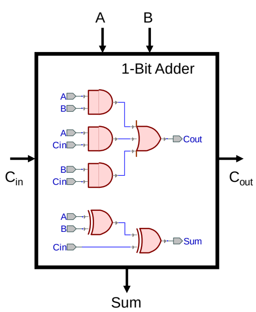
A	B	Cin	Cout	S
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

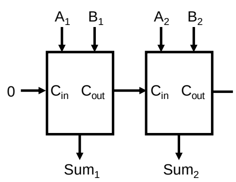


Cout = B Cin + A Cin + A B

S = A’ B’ Cin + A’ B Cin’ + A B’ Cin’ + A B Cin
  = A’ (B’ Cin + B Cin’) + A (B’ Cin’ + B Cin)
  = A’ Z + A Z’
  = A xor Z = A xor (B xor Cin)

Preview: A 2-bit ripple-carry adder





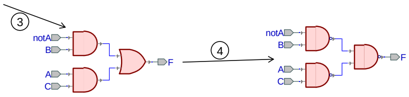
Mapping truth tables to logic gates

- Given a truth table:
  1. Write the Boolean expression
  2. Minimize the Boolean expression
  3. Draw as gates
  4. Map to available gates

A	B	C	F
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

1

F = A'BC'+A'BC+AB'C+ABC
2
  = A'B(C'+C)+AC(B'+B)
  = A'B+AC



Canonical forms

- Truth table is the unique signature of a Boolean function
- The same truth table can have many gate realizations
  - we've seen this already
  - depends on how good we are at Boolean simplification
- Canonical forms
  - standard forms for a Boolean expression
  - we all come up with the same expression

Sum-of-products canonical forms

- Also known as disjunctive normal form
- Also known as minterm expansion

F = *001* *011* *101* *110* *111*

F = A'B'C + A'BC + AB'C + ABC' + ABC

A	B	C	F	F'
0	0	0	0	1
0	0	1	1	0
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	0

F' = A'B'C' + A'BC' + AB'C'

Sum-of-products canonical form (cont'd)

- Product term (or minterm)
  - ANDed product of literals – input combination for which output is true
  - each variable appears exactly once, true or inverted (but not both)

A	B	C	minterms	
0	0	0	A'B'C'	m0
0	0	1	A'B'C	m1
0	1	0	A'BC'	m2
0	1	1	A'BC	m3
1	0	0	AB'C'	m4
1	0	1	AB'C	m5
1	1	0	ABC'	m6
1	1	1	ABC	m7

short-hand notation for minterms of 3 variables

F in canonical form:

F(A, B, C) = Σm(1,3,5,6,7)
= m1 + m3 + m5 + m6 + m7
= A'B'C + A'BC + AB'C + ABC' + ABC

canonical form $\neq$ minimal form

F(A, B, C) = A'B'C + A'BC + AB'C + ABC + ABC'
= (A'B' + A'B + AB' + AB)C + ABC'
= ((A' + A)(B' + B))C + ABC'
= C + ABC'
= ABC' + C
= AB + C

Product-of-sums canonical form

- Also known as conjunctive normal form
- Also known as maxterm expansion

F = *000* *010* *100*

F = (A + B + C) (A + B' + C) (A' + B + C)

A	B	C	F	F'
0	0	0	0	1
0	0	1	1	0
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	0

F' = (A + B + C') (A + B' + C') (A' + B + C') (A' + B' + C) (A' + B' + C')

Product-of-sums canonical form (cont'd)

- Sum term (or maxterm)
  - ORed sum of literals – input combination for which output is false
  - each variable appears exactly once, true or inverted (but not both)

A	B	C	maxterms	
0	0	0	A+B+C	M0
0	0	1	A+B+C'	M1
0	1	0	A+B'+C	M2
0	1	1	A+B'+C'	M3
1	0	0	A'+B+C	M4
1	0	1	A'+B+C'	M5
1	1	0	A'+B'+C	M6
1	1	1	A'+B'+C'	M7

short-hand notation for maxterms of 3 variables

F in canonical form:

F(A, B, C) = ΠM(0,2,4)
= M0 • M2 • M4
= (A + B + C) (A + B' + C) (A' + B + C)

canonical form $\neq$ minimal form

F(A, B, C) = (A + B + C) (A + B' + C) (A' + B + C)
= (A + B + C) (A + B' + C)
(A + B + C) (A' + B + C)
= (A + C) (B + C)

S-o-P, P-o-S, and de Morgan's theorem

- Complement of function in sum-of-products form
  - F' = A'B'C' + A'BC' + AB'C'
- Complement again and apply de Morgan's and get the product-of-sums form
  - (F')' = (A'B'C' + A'BC' + AB'C')'
  - F = (A + B + C) (A + B' + C) (A' + B + C)
- Complement of function in product-of-sums form
  - F' = (A + B + C') (A + B' + C') (A' + B + C') (A' + B' + C) (A' + B' + C')
- Complement again and apply de Morgan's and get the sum-of-product form
  - (F')' = ((A + B + C')(A + B' + C')(A' + B + C')(A' + B' + C)(A' + B' + C'))'
  - F = A'B'C + A'BC + AB'C + ABC' + ABC

Predicate Calculus

- *Predicate* or *Propositional Function*
  - A function that returns a truth value
- "x is a cat"
- "x is prime"
- "student x has taken course y"
- "$x > y$"
- "$x + y = z$"

Quantifiers

- $\forall x\, P(x)$: $P(x)$ is true for every x in the domain
- $\exists x\, P(x)$: There is an x in the domain for which $P(x)$ is true

Statements with quantifiers

Domain: Positive Integers

Even(x)
Odd(x)
Prime(x)
Greater(x,y)
Equal(x,y)

- $\exists x$ Even(x)
- $\forall x$ Odd(x)
- $\forall x$ (Even(x) $\vee$ Odd(x))
- $\exists x$ (Even(x) $\wedge$ Odd(x))
- $\forall x$ Greater(x+1, x)
- $\exists x$ (Even(x) $\wedge$ Prime(x))

Statements with quantifiers

Domain: Positive Integers

Even(x)
Odd(x)
Prime(x)
Greater(x,y)
Equal(x,y)

- $\forall x \exists y$ Greater (y, x)
- $\forall x \exists y$ Greater (x, y)
- $\forall x \exists y$ (Greater(y, x) $\wedge$ Prime(y))
- $\forall x$ (Prime(x) $\rightarrow$ (Equal(x, 2) $\vee$ Odd(x))
- $\exists x \exists y$(Equal(x, y + 2) $\wedge$ Prime(x) $\wedge$ Prime(y))

Statements with quantifiers

Domain: Positive Integers

Even(x)
Odd(x)
Prime(x)
Greater(x,y)
Equal(x,y)

- "There is an odd prime"
- "If x is greater than two, x is not an even prime"
- $\forall x \forall y \forall z$ ((Equal(z, x+y) $\wedge$ Odd(x) $\wedge$ Odd(y))$\rightarrow$ Even(z))
- "There exists an odd integer that is the sum of two primes"

Goldbach's Conjecture

- Every even integer greater than two can be expressed as the sum of two primes

Even(x)
Odd(x)
Prime(x)
Greater(x,y)
Equal(x,y)

Domain: Positive Integers