CSE 311 Foundations of Computing I

Autumn 2011
Lecture 2
More Propositional Logic
Application: Circuits
Propositional Equivalence

Administrative

- Course web: http://www.cs.washington.edu/311
  - Homework, Lecture slides, Office Hours ...
- Office Hours: starting today
- Homework:
  - Paper turn-in (stapled) handed in at the **start** of class on due date (Wednesday).
    - No on-line turn-in.
  - Individual.
    - OK to discuss with a couple of others but nothing recorded from discussion and write-up done much later

Administrative

- Coursework and grading
  - Weekly written homework ~ 45-50 %
  - Midterm (November 4) ~ 15-20%
  - Final (December 12) ~ 30-35%
- A note about Extra Credit problems
  - Not required to get a 4.0
    - Recorded separately and grades calculated entirely without it
    - Fact that others do them can't lower your score
    - In total may raise grade by 0.1 (occasionally 0.2)
      - Each problem ends up worth less than required ones

Recall…Connectives

p	$\neg p$
T	F
F	T

NOT

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

AND

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

OR

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

XOR

$p \rightarrow q$

p	q	$p \rightarrow q$

- Implication
  - *p* implies *q*
  - whenever *p* is true *q* must be true
  - if *p* then *q*
  - *q* if *p*
  - *p* is sufficient for *q*
  - *p* only if *q*

"If you behave then I'll buy you ice cream"

- What if you don't behave?

“If pigs can whistle then horses can fly”

Converse, Contrapositive, Inverse

- Implication: $p \rightarrow q$
- Converse: $q \rightarrow p$
- Contrapositive: $\neg q \rightarrow \neg p$
- Inverse: $\neg p \rightarrow \neg q$

- Are these the same?

Example
p: “x is divisible by 2”
q: “x is divisible by 4”

Biconditional $p \leftrightarrow q$

- p iff q
- p is equivalent to q
- p implies q and q implies p

p	q	$p \leftrightarrow q$

English and Logic

- You cannot ride the roller coaster if you are under 4 feet tall unless you are older than 16 years old
  - q: you can ride the roller coaster
  - r: you are under 4 feet tall
  - s: you are older than 16

$(r \wedge \neg s) \rightarrow \neg q$

Digital Circuits

- Computing with logic
  - **T** corresponds to 1 or “high” voltage
  - **F** corresponds to 0 or “low” voltage
- Gates
  - Take inputs and produce outputs
    - Functions
  - Several kinds of gates
  - Correspond to propositional connectives
    - Only symmetric ones (order of inputs irrelevant)

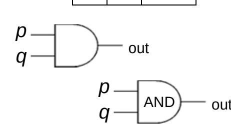
Gates

AND connective
$p \wedge q$

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

AND gate

p	q	out
1	1	1
1	0	0
0	1	0
0	0	0



“block looks like D of AND”

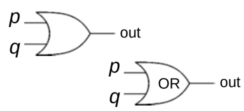
Gates

OR connective
$p \vee q$

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

OR gate

p	q	*out*
1	1	1
1	0	1
0	1	1
0	0	0



"arrowhead block looks like V"

Gates

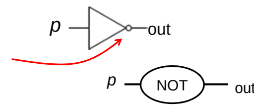
NOT connective
$\neg p$

p	$\neg p$
T	F
F	T

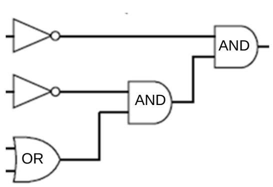
NOT gate
(inverter)

p	*out*
1	0
0	1

Bubble most important
for this diagram

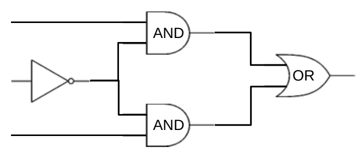


Combinational Logic Circuits



Values get sent along wires connecting gates

Combinational Logic Circuits



Wires can send one value to multiple gates

Logical equivalence

- Terminology: A compound proposition is a
  - *Tautology* if it is always true
  - *Contradiction* if it is always false
  - *Contingency* if it can be either true or false

$p \vee \neg p$

$p \oplus p$

$(p \to q) \wedge p$

$(p \wedge q) \vee (p \wedge \neg q) \vee (\neg p \wedge q) \vee (\neg p \wedge \neg q)$

Logical Equivalence

- p and q are *logically equivalent* iff $p \leftrightarrow q$ is a tautology
  - *i.e.* p and q have the same truth table
- The notation $p \equiv q$ denotes p and q are logically equivalent
- Example: $p \equiv \neg\neg p$

p	$\neg p$	$\neg\neg p$	$p \leftrightarrow \neg\neg p$

De Morgan's Laws

- $\neg (p \wedge q) \equiv \neg p \vee \neg q$
- $\neg (p \vee q) \equiv \neg p \wedge \neg q$

- What are the negations of:
  - The Yankees and the Phillies will play in the World Series
  - It will rain today or it will snow on New Year's Day

De Morgan's Laws

Example: $\neg (p \wedge q) \equiv (\neg p \vee \neg q)$

p	q	$\neg p$	$\neg q$	$\neg p \vee \neg q$	$p \wedge q$	$\neg(p \wedge q)$	$\neg(p \wedge q) \leftrightarrow (\neg p \vee \neg q)$
T	T						
T	F						
F	T						
F	F						

Law of Implication

Example: $(p \rightarrow q) \equiv (\neg p \vee q)$

p	q	$p \rightarrow q$	$\neg p$	$\neg p \vee q$	$(p \rightarrow q) \leftrightarrow (\neg p \vee q)$

Computing equivalence

- Describe an algorithm for computing if two logical expressions/circuits are equivalent
- What is the run time of the algorithm?

Understanding connectives

- Reflect basic rules of reasoning and logic
- Allow manipulation of logical formulas
  - Simplification
  - Testing for equivalence
- Applications
  - Query optimization
  - Search optimization and caching
  - Artificial Intelligence
  - Program verification

Properties of logical connectives

- Identity
- Domination
- Idempotent
- Commutative
- Associative
- Distributive
- Absorption
- Negation

Equivalences relating to implication

- $p \to q \equiv \neg p \lor q$
- $p \to q \equiv \neg q \to \neg p$
- $p \lor q \equiv \neg p \to q$
- $p \land q \equiv \neg (p \to \neg q)$
- $p \leftrightarrow q \equiv (p \to q) \land (q \to p)$
- $p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
- $p \leftrightarrow q \equiv (p \land q) \lor (\neg p \land \neg q)$
- $\neg (p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Logical Proofs

- To show P is equivalent to Q
  – Apply a series of logical equivalences to subexpressions to convert P to Q
- To show P is a tautology
  – Apply a series of logical equivalences to subexpressions to convert P to **T**

Show $(p \land q) \to (p \lor q)$ is a tautology

Show $(p \to q) \to r$ and $p \to (q \to r)$ are not equivalent