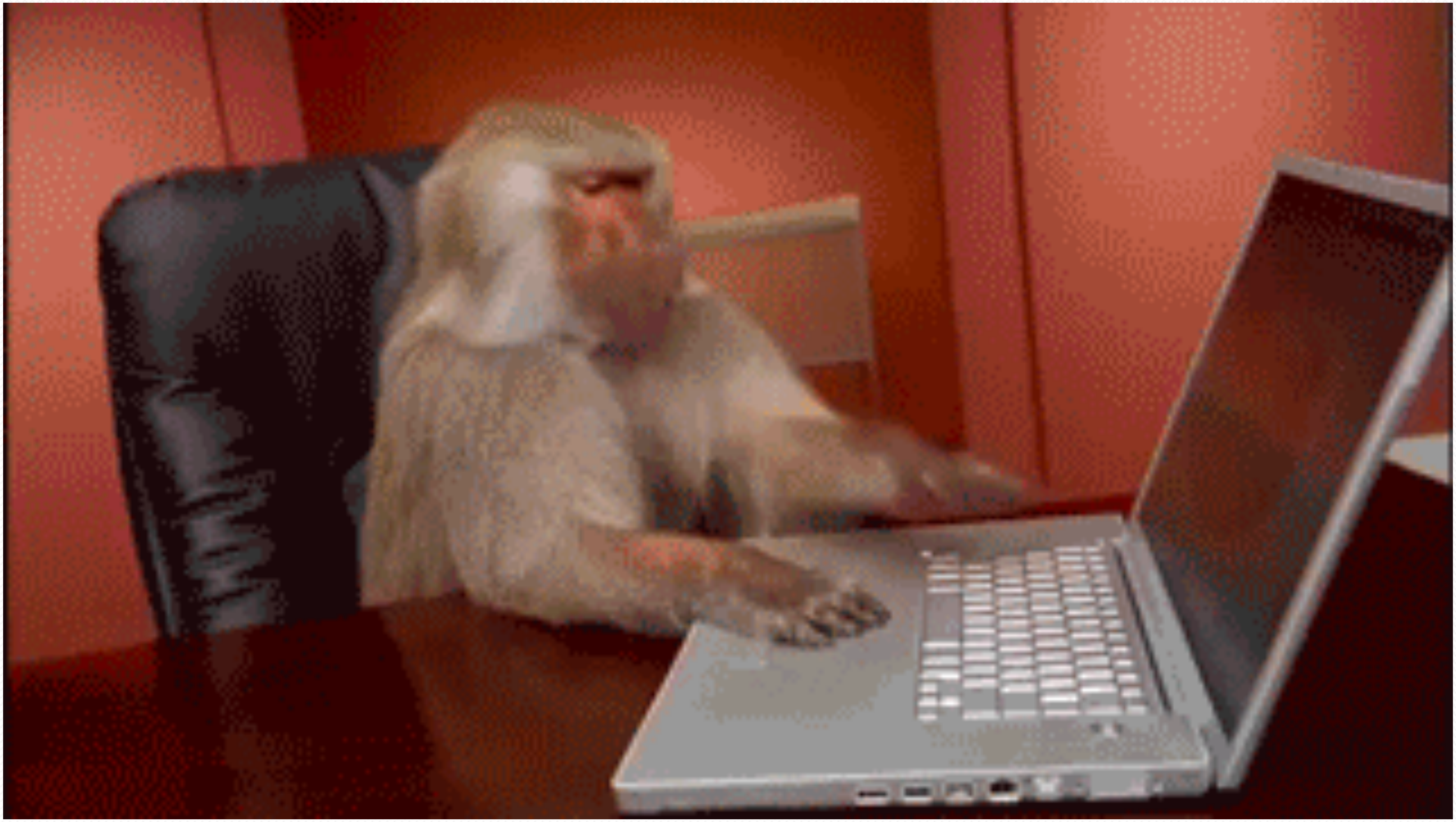
# Building Java Programs

Chapter 13

binary search and complexity

**reading: 13.1-13.2**



# Tips for testing

- You cannot test every possible input, parameter value, etc.
  - Think of a limited set of tests likely to expose bugs.

- Think about boundary cases
  - Positive; zero; negative numbers
  - Right at the edge of an array or collection's size

- Think about empty cases and error cases
  - 0, -1, null;  an empty list or array

- test behavior in combination
  - Maybe `add` usually works, but fails after you call `remove`
  - Make multiple calls;  maybe `size` fails the second time only

# Searching methods

- Implement the following methods:
  - `indexOf` – returns first index of element, or -1 if not found
  - `contains` - returns true if the list contains the given int value

- Why do we need `isEmpty` and `contains` when we already have `indexOf` and `size` ?
  - Adds convenience to the client of our class:

```
// less elegant                        // more elegant
if (myList.size() == 0) {              if (myList.isEmpty()) {
if (myList.indexOf(42) >= 0) {         if (myList.contains(42)) {
```

# Sequential search

- **sequential search**: Locates a target value in an array / list by examining each element from start to finish. Used in `indexOf`.
  - How many elements will it need to examine?
  - Example: Searching the array below for the value **42**:

| index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| value | -4 | 2 | 7 | 10 | 15 | 20 | 22 | 25 | 30 | 36 | 42 | 50 | 56 | 68 | 85 | 92 | 103 |

i

  - The array is sorted. Could we take advantage of this?

# Binary search (13.1)

- **binary search**: Locates a target value in a *sorted* array or list by successively eliminating half of the array from consideration.
  - How many elements will it need to examine?
  - Example: Searching the array below for the value **42**:

| index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| value | -4 | 2 | 7 | 10 | 15 | 20 | 22 | 25 | 30 | 36 | 42 | 50 | 56 | 68 | 85 | 92 | 103 |

min mid max

# Arrays.binarySearch

```
// searches an entire sorted array for a given value
// returns its index if found;  a negative number if not found
// Precondition: array is sorted
Arrays.binarySearch(array, value)

// searches given portion of a sorted array for a given value
// examines minIndex (inclusive) through maxIndex (exclusive)
// returns its index if found;  a negative number if not found
// Precondition: array is sorted
Arrays.binarySearch(array, minIndex, maxIndex, value)
```

- The `binarySearch` method in the `Arrays` class searches an array very efficiently if the array is sorted.
  - You can search the entire array, or just a range of indexes (useful for "unfilled" arrays such as the one in `ArrayIntList`)

# Using `binarySearch`

```
// index     0   1   2   3    4    5    6    7    8    9   10   11   12   13   14   15
int[] a = {-4, 2, 7, 9, 15, 19, 25, 28, 30, 36, 42, 50, 56, 68, 85, 92};

int index  = Arrays.binarySearch(a, 0, 16, 42);    // index1 is 10
int index2 = Arrays.binarySearch(a, 0, 16, 21);    // index2 is -7
```

- `binarySearch` returns the index where the value is found
- if the value is *not* found, `binarySearch` returns:

  `-(insertionPoint + 1)`

  - where `insertionPoint` is the index where the element *would* have been, if it had been in the array in sorted order.
  - To insert the value into the array, negate `insertionPoint` + 1

```
int indexToInsert21 = -(index2 + 1);  // 6
```

# Runtime Efficiency (13.2)

- How much better is binary search than sequential search?

- **efficiency**: measure of computing resources used by code.
  - can be relative to speed (time), memory (space), etc.
  - most commonly refers to run time

- Assume the following:
  - Any single Java statement takes same amount of time to run.
  - A method call's runtime is measured by the total of the statements inside the method's body.
  - A loop's runtime, if the loop repeats N times, is N times the runtime of the statements in its body.

# Efficiency examples

```
statement1;
statement2;
statement3;

for (int i = 1; i <= N; i++) {
    statement4;
}

for (int i = 1; i <= N; i++) {
    statement5;
    statement6;
    statement7;
}
```

3

N

3N

4N + 3

# Efficiency examples 2

```
for (int i = 1; i <= N; i++) {
    for (int j = 1; j <= N; j++) {
        statement1;
    }
}

for (int i = 1; i <= N; i++) {
    statement2;
    statement3;
    statement4;
    statement5;
}
```

$N^2$

4N

$N^2 + 4N$

- How many statements will execute if N = 10? If N = 1000?

# Algorithm growth rates (13.2)

- We measure runtime in proportion to the input data size, N.
  - **growth rate**: Change in runtime as N changes.

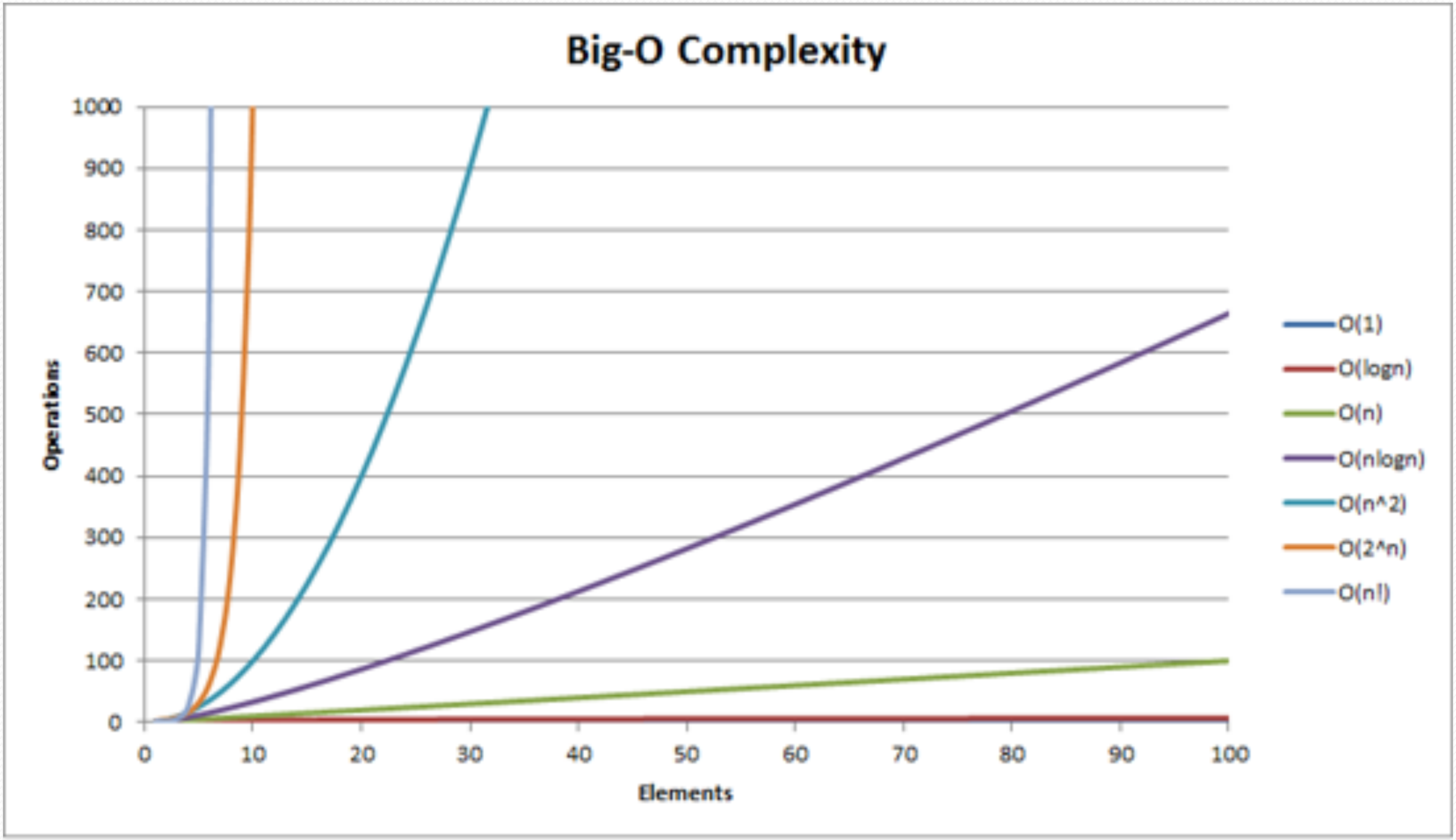
- Say an algorithm runs $\mathbf{0.4N^3 + 25N^2 + 8N + 17}$ statements.
  - Consider the runtime when N is *extremely large* .
  - We ignore constants like 25 because they are tiny next to N.
  - The highest-order term ($N^3$) dominates the overall runtime.
  - We say that this algorithm runs "on the order of" $N^3$.
  - or $\mathbf{O(N^3)}$ for short ("**Big-Oh** of N cubed")

# Complexity classes

- **complexity class**: A category of algorithm efficiency based on the algorithm's relationship to the input size N.

| Class | Big-Oh | If you double N, ... | Example |
|---|---|---|---|
| constant | O(1) | unchanged | 10ms |
| logarithmic | $O(\log_2 N)$ | increases slightly | 175ms |
| linear | O(N) | doubles | 3.2 sec |
| log-linear | $O(N \log_2 N)$ | slightly more than doubles | 6 sec |
| quadratic | $O(N^2)$ | quadruples | 1 min 42 sec |
| cubic | $O(N^3)$ | multiplies by 8 | 55 min |
| ... | ... | ... | ... |
| exponential | $O(2^N)$ | multiplies drastically | $5 * 10^{61}$ years |

# Complexity classes




http://recursive-design.com/blog/2010/12/07/comp-sci-101-big-o-notation/ - post about a Google interview

# Sequential search

- What is its complexity class?

```
public int indexOf(int value) {
    for (int i = 0; i < size; i++) {
        if (elementData[i] == value) {
            return i;
        }
    }
    return -1;    // not found
}
```

N

| index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| value | -4 | 2 | 7 | 10 | 15 | 20 | 22 | 25 | 30 | 36 | 42 | 50 | 56 | 68 | 85 | 92 | 103 |

- On average, "only" N/2 elements are visited
  - 1/2 is a constant that can be ignored

# Collection efficiency

- Efficiency of our `ArrayIntList` or Java's `ArrayList`:

| **Method** | **ArrayList** |
|---|---|
| `add` | O(1) |
| `add(`**index**`, `**value**`)` | O(N) |
| `indexOf` | O(N) |
| `get` | O(1) |
| `remove` | O(N) |
| `set` | O(1) |
| `size` | O(1) |

# Binary search

- **binary search** successively eliminates half of the elements.
  - *Algorithm:* Examine the middle element of the array.
    - If it is too big, eliminate the right half of the array and repeat.
    - If it is too small, eliminate the left half of the array and repeat.
    - Else it is the value we're searching for, so stop.

  - Which indexes does the algorithm examine to find value **42**?
  - What is the runtime complexity class of binary search?

| index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| value | -4 | 2 | 7 | 10 | 15 | 20 | 22 | 25 | 30 | 36 | 42 | 50 | 56 | 68 | 85 | 92 | 103 |

min

mid

max

# Binary search runtime

- For an array of size N, it eliminates ½ until 1 element remains.

  N, N/2, N/4, N/8, ..., 4, 2, 1

  - How many divisions does it take?

- Think of it from the other direction:
  - How many times do I have to multiply by 2 to reach N?

    1, 2, 4, 8, ..., N/4, N/2, N
  - Call this number of multiplications "x".

  $2^x = N$

  $\mathbf{x = \log_2 N}$

- Binary search is in the **logarithmic** complexity class.

# Range algorithm

What complexity class is this algorithm? Can it be improved?

```
// returns the range of values in the given array;
// the difference between elements furthest apart
// example: range({17, 29, 11, 4, 20, 8}) is 25
public static int range(int[] numbers) {
    int maxDiff = 0;     // look at each pair of values
    for (int i = 0; i < numbers.length; i++) {
        for (int j = 0; j < numbers.length; j++) {
            int diff = Math.abs(numbers[j] - numbers[i]);
            if (diff > maxDiff) {
                maxDiff = diff;
            }
        }
    }
    return diff;
}
```

# Range algorithm

What complexity class is this algorithm? Can it be improved?

```
// returns the range of values in the given array;
// the difference between elements furthest apart
// example: range({17, 29, 11, 4, 20, 8}) is 25
public static int range(int[] numbers) {
    int maxDiff = 0;     // look at each pair of values
    for (int i = 0; i < numbers.length; i++) {
        for (int j = 0; j < numbers.length; j++) {
            int diff = Math.abs(numbers[j] - numbers[i]);
            if (diff > maxDiff) {
                maxDiff = diff;
            }
        }
    }
    return diff;
}
```

# Range algorithm 2

The last algorithm is **O(N²)**.  A slightly better version:

```
// returns the range of values in the given array;
// the difference between elements furthest apart
// example: range({17, 29, 11, 4, 20, 8}) is 25
public static int range(int[] numbers) {
    int maxDiff = 0;     // look at each pair of values
    for (int i = 0; i < numbers.length; i++) {
        for (int j = i + 1; j < numbers.length; j++) {
            int diff = Math.abs(numbers[j] - numbers[i]);
            if (diff > maxDiff) {
                maxDiff = diff;
            }
        }
    }
    return diff;
}
```

# Range algorithm 3

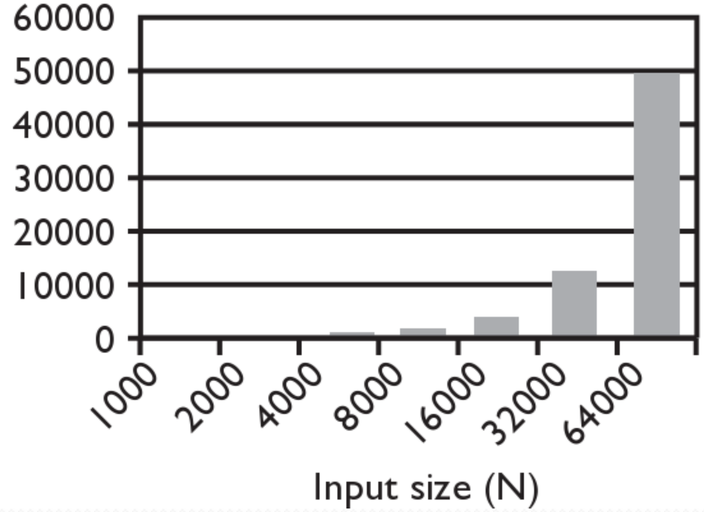
This final version is **O(N)**.  It runs MUCH faster:

```
// returns the range of values in the given array;
// example: range({17, 29, 11, 4, 20, 8}) is 25
public static int range(int[] numbers) {
    int max = numbers[0];     // find max/min values
    int min = max;
    for (int i = 1; i < numbers.length; i++) {
        if (numbers[i] < min) {
            min = numbers[i];
        }
        if (numbers[i] > max) {
            max = numbers[i];
        }
    }
    return max - min;
}
```

# Runtime of first 2 versions

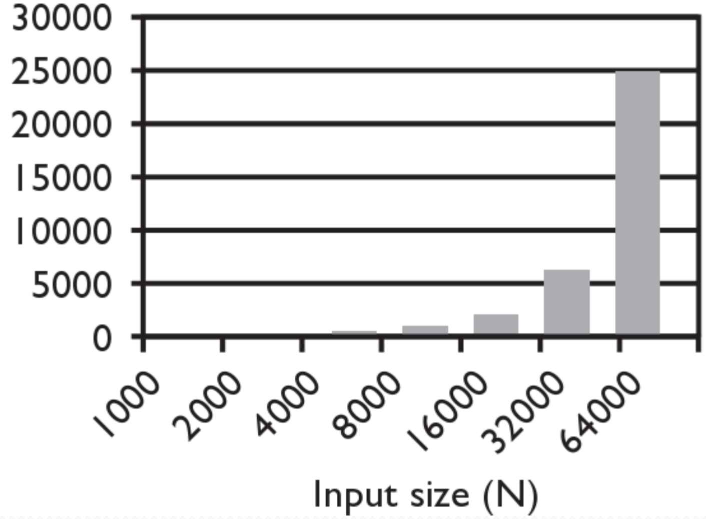
- Version 1:

| N | Runtime (ms) |
|---|---|
| 1000 | 15 |
| 2000 | 47 |
| 4000 | 203 |
| 8000 | 781 |
| 16000 | 3110 |
| 32000 | 12563 |
| 64000 | 49937 |




- Version 2:

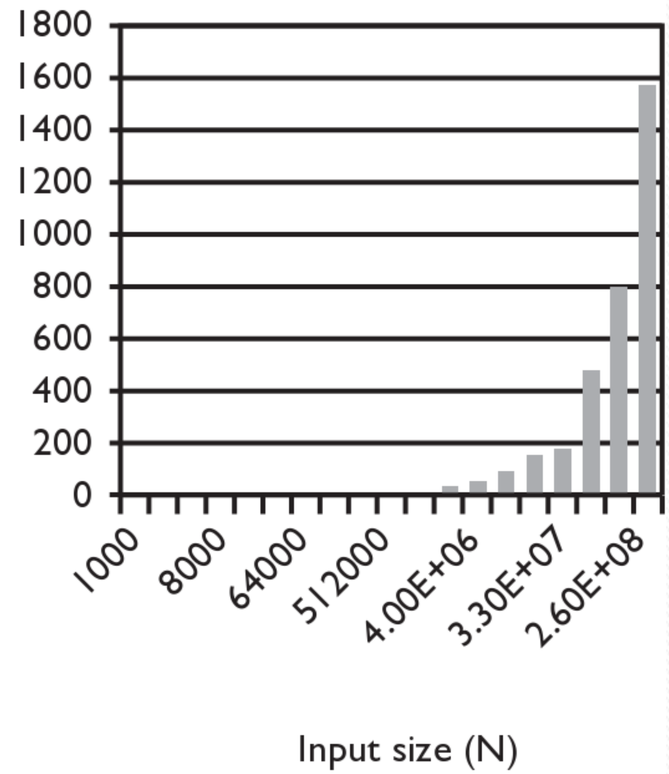
| N | Runtime (ms) |
|---|---|
| 1000 | 16 |
| 2000 | 16 |
| 4000 | 110 |
| 8000 | 406 |
| 16000 | 1578 |
| 32000 | 6265 |
| 64000 | 25031 |



# Runtime of 3rd version

- Version 3:

| N | Runtime (ms) |
|---|---|
| 1000 | 0 |
| 2000 | 0 |
| 4000 | 0 |
| 8000 | 0 |
| 16000 | 0 |
| 32000 | 0 |
| 64000 | 0 |
| 128000 | 0 |
| 256000 | 0 |
| 512000 | 0 |
| 1e6 | 0 |
| 2e6 | 16 |
| 4e6 | 31 |
| 8e6 | 47 |
| 1.67e7 | 94 |
| 3.3e7 | 188 |
| 6.5e7 | 453 |
| 1.3e8 | 797 |
| 2.6e8 | 1578 |



# Max subsequence sum

- Write a method `maxSum` to find the largest sum of any contiguous subsequence in an array of integers.
  - Easy for all positives: include the whole array.
  - What if there are negatives?

| index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| value | 2 | 1 | -4 | 10 | 15 | -2 | 22 | -8 | 5 |

Largest sum: 10 + 15 + -2 + 22 = 45

  - (Let's define the max to be 0 if the array is entirely negative.)

- Ideas for algorithms?

# Algorithm 1 pseudocode

```
maxSum(a):
    max = 0.
    for each starting index i:
        for each ending index j:
            sum = add the elements from a[i] to a[j].
            if sum > max,
                max = sum.

    return max.
```

| index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| value | 2 | 1 | -4 | 10 | 15 | -2 | 22 | -8 | 5 |

# Algorithm 1 code

- What complexity class is this algorithm?
  - **O(N$^3$)**. Takes a few seconds to process 2000 elements.

```
public static int maxSum1(int[] a) {
    int max = 0;
    for (int i = 0; i < a.length; i++) {
        for (int j = i; j < a.length; j++) {
            // sum = add the elements from a[i] to a[j].
            int sum = 0;
            for (int k = i; k <= j; k++) {
                sum += a[k];
            }
            if (sum > max) {
                max = sum;
            }
        }
    }
    return max;
}
```

# Flaws in algorithm 1

- Observation: We are redundantly re-computing sums.
  - For example, we compute the sum between indexes 2 and 5:
    a[2] + a[3] + a[4] + a[5]
  - Next we compute the sum between indexes 2 and 6:
    a[2] + a[3] + a[4] + a[5] + a[6]
  - We already had computed the sum of 2-5, but we compute it again as part of the 2-6 computation.
  - Let's write an improved version that avoids this flaw.

| index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| value | 2 | 1 | -4 | 10 | 15 | -2 | 22 | -8 | 5 |

# Algorithm 2 code

- What complexity class is this algorithm?
  - **O(N²)**. Can process tens of thousands of elements per second.

```
public static int maxSum2(int[] a) {
    int max = 0;
    for (int i = 0; i < a.length; i++) {
        int sum = 0;
        for (int j = i; j < a.length; j++) {
            sum += a[j];
            if (sum > max) {
                max = sum;
            }
        }
    }
    return max;
}
```

| index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| value | 2 | 1 | -4 | 10 | 15 | -2 | 22 | -8 | 5 |

# A clever solution

- *Claim 1* : A max range cannot start with a negative-sum range.

| i ... j | j+1 ... k |
|---|---|
| < 0 | sum(j+1, k) |
| sum(i, k) < sum(j+1, k) | |

- *Claim 2* : If sum(i, j-1) ≥ 0 and sum(i, j) < 0, any max range that ends at j+1 or higher cannot start at any of i through j.

| i ... j-1 | j | j+1 ... k |
|---|---|---|
| ≥ 0 | < 0 | sum(j+1, k) |
| < 0 | | sum(j+1, k) |
| | | sum(?, k) < sum(j+1, k) |

  - Together, these observations lead to a very clever algorithm...

# Algorithm 3 code

- What complexity class is this algorithm?
  - **O(N).** Handles many millions of elements per second!

```
public static int maxSum3(int[] a) {
    int max = 0;
    int sum = 0;
    int i = 0;
    for (int j = 0; j < a.length; j++) {
        if (sum < 0) {    // if sum becomes negative, max range
            i = j;        // cannot start with any of i - j-1
            sum = 0;      // (Claim 2)
        }
        sum += a[j];
        if (sum > max) {
            max = sum;
        }
    }
    return max;
}
```