## CSE 143 Java

Program Efficiency &
Introduction to Complexity Theory



## GREAT IDEAS IN COMPUTER SCIENCE

**ANALYSIS OF ALGORITHMIC COMPLEXITY**



## Overview

- Topics
  - Measuring time and space used by algorithms
  - Machine-independent measurements
  - Costs of operations
  - Comparing algorithms
  - Asymptotic complexity – O( ) notation and complexity classes
- Reading:
  - Textbook: Ch. 21



## Comparing Algorithms

- Example: We've seen two different list implementations
  - Dynamic expanding array
  - Linked list
- Which is "better"?
- How do we measure?
  - Stopwatch? Why or why not?

## Program Efficiency & Resources

- Goal: Find way to measure "resource" usage in a way that is independent of particular machines/implementations
- Resources
  - Execution time
  - Execution space
  - Network bandwidth
  - others
- We will focus on execution time
  - Basic techniques/vocabulary apply to other resource measures



## Example

- What is the running time of the following method?

```
// Return the sum of the elements in array.
double sum(double[ ] rainMeas) {
    double ans = 0.0;
    for (int k = 0; k < rainMeas.length; k++) {
        ans = ans + rainMeas[k];
    }
    return ans;
}
```

- How do we analyze this?



## Analysis of Execution Time

1. First: describe the *size* of the problem in terms of one or more parameters
   - For sum, size of array makes sense
   - Often size of data structure, but can be magnitude of some numeric parameter, etc.
2. Then, count the number of steps needed as a function of the problem size

- Need to define what a "step" is.
  - First approximation: one simple statement
  - More complex statements will be multiple steps



## Cost of operations: Constant Time Ops

- Constant-time operations: each take one abstract time “step”
  - Simple variable declaration/initialization (double sum = 0.0;)
  - Assignment of numeric or reference values (var = value;)
  - Arithmetic operation (+, -, *, /, %)
  - Array subscripting (a[index])
  - Simple conditional tests (x < y, p != null)
  - Operator *new* itself (not including constructor cost)

    Note: *new* takes significantly longer than simple arithmetic or assignment, but its cost is independent of the problem we're trying to analyze
- Note: watch out for things like method calls or constructor invocations that look simple, but are expensive

## Cost of operations: Zero-time Ops

- Compiler can sometimes pay the whole cost of setting up operations
  - Nothing left to do at runtime
- Variable declarations without initialization
  double[ ] overdrafts;
- Variable declarations with compile-time constant initializers
  static final int maxButtons = 3;
- Casts (of reference types, at least)
  ... (Double) checkBalance



## Sequences of Statements

- Cost of
  S1; S2; … Sn
  is sum of the costs of S1 + S2 + … + Sn



## Conditional Statements

- The two branches of an if-statement might take different times. What to do??

```
if (condition) {
    S1;
} else {
    S2;
}
```

- Hint: Depends on analysis goals
  - "Worst case": the longest it could possibly take, under any circumstances
  - "Average case": the expected or average number of steps
  - "Best case": the shortest possible number of steps, under some special circumstance
- Generally, worst case is most important to analyze



## Analyzing Loops

- Basic analysis
  1. Calculate cost of each iteration
  2. Calculate number of iterations
  3. Total cost is the product of these

  Caution -- sometimes need to add up the costs differently if cost of each iteration is not roughly the same
- Nested loops
  - Total cost is number of iterations or the outer loop times the cost of the inner loop
  - same caution as above

## Method Calls

- Cost for calling a function is cost of...

  cost of **evaluating** the arguments (constant or non-constant)

  + cost of actually **calling** the function (constant overhead)

  + cost of **passing** each parameter (normally constant time in Java for both numeric and reference values)

  + cost of **executing** the function body (constant or non-constant?)

```
System.out.print(this.lineNumber);
System.out.println("Answer is " + Math.sqrt(3.14159));
```

- Terminology note: "evaluating" and "passing" an argument are two different things!



## Exact Complexity Function

- Careful analysis of an algorithm leads to an algebraic formula
- The "exact complexity function" gives the number of steps as a function of the problem size
- What can we do with it:
  - Predict running time in a particular case (given n, given type of computer)?
  - Predict comparative running times for two different n (on same type of computer)?
  - ***** Get a general feel for the potential performance of an algorithm
  - ***** Compare predicted running time of two different algorithms for the same problem (given same n)



## A Graph is Worth A Bunch of Words

- Graphs are a good tool to illustrate, study, and compare complexity functions

- Fun math review for you
  - How do you graph a function?
  - What are the shapes of some common functions? For example, ones mentioned in these slides or the textbook.



## Exercise

- Analyze the running time of printMultTable
  - Pick the problem size
  - Count the number of steps

```
// print multiplication table with
// n rows and columns
void printMultTable(int n) {
    for (int k=0; k <=n; k++) {
        printRow(k, n);
    }
}
```

```
// print row r with length n of a
// multiplication table
void printRow(int r, int n) {
    for (int k = 0; k <= r; k++) {
        System.out.print(r*k + " ");
    }
    System.out.println( );
}
```

## Comparing Algorithms

- Suppose we analyze two algorithms and get these times (numbers of steps):
  - Algorithm 1: $37n + 2n^2 + 120$
  - Algorithm 2: $50n + 42$

  How do we compare these? What really matters?
- Answer: In the long run, the thing that is most interesting is the cost as the problem size n gets large
  - What are the costs for n=10, n=100; n=1,000; n=1,000,000?
  - Computers are so fast that how long it takes to solve small problems is rarely of interest



## Orders of Growth

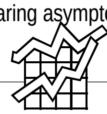
- Examples:

| N | $\log_2 N$ | 5N | $N \log_2 N$ | $N^2$ | $2^N$ |
|---|---|---|---|---|---|
| **8** | 3 | 40 | 24 | 64 | 256 |
| **16** | 4 | 80 | 64 | 256 | 65536 |
| **32** | 5 | 160 | 160 | 1024 | $\sim 10^9$ |
| **64** | 6 | 320 | 384 | 4096 | $\sim 10^{19}$ |
| **128** | 7 | 640 | 896 | 16384 | $\sim 10^{38}$ |
| **256** | 8 | 1280 | 2048 | 65536 | $\sim 10^{76}$ |
| 10000 | **13** | **50000** | **$10^5$** | **$10^8$** | **$\sim 10^{3010}$** |



## Asymptotic Complexity

- Asymptotic: Behavior of complexity function as problem size gets large
  - Only thing that really matters is higher-order term
  - Can drop low order terms and constants
- The asymptotic complexity gives us a (partial) way to answer "which algorithm is more efficient"
  - Algorithm 1: $37n + 2n^2 + 120$ is proportional to $n^2$
  - Algorithm 2: $50n + 42$ is proportional to n
- Graphs of functions are handy tool for comparing asymptotic behavior





## Big-O Notation

- Definition: If f(n) and g(n) are two complexity functions, we say that

  $f(n) = O(g(n))$ ( pronounced f(n) is O(g(n)) or is order g(n) )

  if there is a constant c such that

  $f(n) \le c \bullet g(n)$

  for all sufficiently large n

## Exercises

- Prove that 5n+3 is O(n)
- Prove that $5n^2 + 42n + 17$ is $O(n^2)$



## Implications

- The notation f(n) = O(g(n)) is *not* an equality
- Think of it as shorthand for
  - "f(n) grows at most like g(n)" or
  - "f grows no faster than g" or
  - "f is bounded by g"
- O( ) notation is a *worst-case* analysis
  - Generally useful in practice
  - Sometimes want *average-case* or *expected-time* analysis if worst-case behavior is not typical (but often harder to analyze)



## Complexity Classes

- Several common complexity classes (problem size n)
  - Constant time: O(k) or O(1)
  - Logarithmic time: O(log n) [Base doesn't matter. Why?]
  - Linear time: O(n)
  - "n log n" time: O(n log n)
  - Quadratic time: $O(n^2)$
  - Cubic time: $O(n^3)$
  - …
  - Exponential time: $O(k^n)$
- $O(n^k)$ is often called *polynomial time*



## Rule of Thumb

- If the algorithm has polynomial time or better: practical
  - typical pattern: examining all data, a fixed number of times
- If the algorithm has exponential time: impractical
  - typical pattern: examine ***all combinations*** of data
- What to do if the algorithm is exponential?
  - Try to find a different algorithm
  - Some problems can be proved not to have a polynomial solution
  - Other problems don't have known polynomial solutions, despite years of study and effort.
  - Sometimes you settle for an approximation:
    - The correct answer most of the time, or
    - An almost-correct answer all of the time

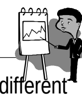
## Big-O Arithmetic

- For most commonly occurring functions, comparison can be enormously simplified with a few simple rules of thumb.
- Memorize complexity classes in order from smallest to largest: O(1), O(log n), O(n), O(n log n), O($n^2$), etc.
- Ignore constant factors
  - $300n + 5n^4 + 6 + 2^n = O(n + n^4 + 2^n)$
- Ignore all but highest order term
  - $O(n + n^4 + 2^n) = O(2^n)$





## Analyzing List Operations (1)



- We can use O( ) notation to compare the costs of different list implementations
- Operation Dynamic Array Linked List
  - Construct empty list
  - Size of the list
  - isEmpty
  - clear





## Analyzing List Operations (2)

- Operation Dynamic Array Linked List
  - Add item to end of list
  - Locate item (contains, indexOf)
  - Add or remove item once it has been located





## Wait! Isn't this totally bogus??

- Write better code!!
  - More clever hacking in the inner loops
    - (assembly language, special-purpose hardware in extreme cases)
- Moore's law: Speeds double every 18 months
  - Wait and buy a faster computer in a year or two!

- But …

## How long is a Computer-Day?

- If a program needs f(n) microseconds to solve some problem, what is the largest single problem it can solve in one full day?
- One day = 1,000,000*24*60*60 = $10^6$*24*36*$10^2$ = $10^6$*25*36*$10^2$ = $10^6$*900*$10^2$ = **$9*10^9$**
- To calculate, set f(n) = **$9*10^9$** and solve for n in each case

| f(n) | n such that f(n) = one day |
|---|---|
| n | $9 * 10^{10}$ |
| 5n | $2.5 * 10^{10}$ |
| n $\log_2 n$ | $3 * 10^9$ |
| $n^2$ | $3 * 10^5$ |
| $n^3$ | $4 * 10^3$ |
| $2^n$ | 36 |



## Speed Up The Computer by 1,000,000

- Suppose technology advances so that a future computer is 1,000,000 fast than today's.
- In one day there are now = **$9*10^9*10^3$** ticks available
- To calculate, set f(n) = **$9*10^{9+3}$** and solve for n in each case

| f(n) | original n for one day | **new n for one day** |
|---|---|---|
| n | $9 * 10^{10}$ | ?????????? |
| 5n | $2.5 * 10^{10}$ | ?????????? |
| n $\log_2 n$ | $3 * 10^9$ | etc. |
| $n^2$ | $3 * 10^5$ | |
| $n^3$ | $4 * 10^3$ | |
| $2^n$ | 36 | |



## How Much Does 1,000,000-faster Buy?

- Divide the new max n by the old max n, to see how much more we can do in a day

| f(n) | n for 1 day | million x, n for 1 day |
|---|---|---|
| n | $9 \times 10^{10}$ | million times larger |
| 5n | $2 \times 10^{10}$ | million times larger |
| n $\log_2 n$ | $3 \times 10^9$ | 60,000 times larger |
| $n^2$ | $3 \times 10^5$ | 1,000 times larger |
| $n^3$ | $4 \times 10^3$ | **100 times larger** |
| $2^n$ | 36 | **+20 larger** |



## Practical Advice For Speed Lovers

- First pick the right algorithm and data structure
  - Implement it carefully, insuring correctness
- Then optimize for speed – but only where it matters
  - Constants do matter in the real world
  - Clever coding can speed things up, but result can be harder to read, modify
- Current state-of-the-art approach: Use measurement tools to find hotspots, then tweak those spots.

"Premature optimization is the root of all evil" – Donald Knuth

> *"It is easier to make a correct program efficient than to make an efficient program correct"*
>
> – Edsgar Dijkstra



## Summary

- Analyze algorithm sufficiently to determine complexity
- Compare algorithms by comparing asymptotic complexity
- For large problems an asymptotically faster algorithm will always trump clever coding tricks

> *"Premature optimization is the root of all evil"*
>
> – Donald Knuth



## Computer Science Note

- Algorithmic complexity theory is one of the key intellectual contributions of Computer Science
- Typical problems
  - What is the worst/average/best-case performance of an algorithm?
  - What is the best complexity bound for all algorithms that solve a particular problem?
- Interesting and (in many cases) complex, sophisticated math
  - Probabilistic and statistical as well as discrete
- Still some key open problems
  - Most notorious: P ?= NP