CSE 143

Program Efficiency

[Chapter 9, pp. 390-401]

8/7/00 U-1

Does it matter?

- Yes! Faster is better
  - Assuming correctness, etc.
- Considerations other than speed
  - Development time
  - Ease of maintenance
  - Extensibility
- Approaches to improving performance
  - Tighter code
  - Better algorithm
  - Better data structure

8/7/00 U-2

Fast code

- C/C++ language/culture encourages tricky coding, often in the name of "efficiency"

```
while (*q++ = *p++) ;
```

- Reasons for caution
  - Correctness
  - Code used by others
  - No need to do compiler's job
  - 90/10 principle (some would say 80/20)

8/7/00 U-3

Measuring Efficiency

- Usually means "time" (to run) or "space" (memory used)
- One way of measuring efficiency is to run the program

```
see how long it takes
see how much memory it uses
```

- Lots of variability when running the program
  - What input data?
  - What hardware platform?
  - What compiler? What compiler options?
- Just because one program runs faster than another right now, will it always be faster?

8/7/00 U-4

Complexity Analysis

- Lots of little details that we'll avoid, to achieve platform-independence
- Use an abstract machine that uses *steps* of time and *units* of memory, instead of seconds or bytes
  - Each elementary operation takes 1 step
  - Each elementary instance occupies 1 unit of memory
- Measure time and space in terms of the *size* of the input rather than details of the specific input
- Allows us to focus on big issues

8/7/00 U-5

Example For Analysis

```
// Input: int A[N], array of N integers
// Output: Sum of all numbers in array A

int Sum(int A[], int N) {
  int sum = 0;
  for ( int j = 0; j < N; j++ )
    sum = sum + A[j];
  return sum;
}
```

How should we analyze this?

8/7/00 U-6

Analysis of `Sum`

- First, describe the *size* of the input in terms of one or more parameters
  - Input to `Sum` is an array of N ints, so size is N.
- Then, count how many steps are used for an input of that size
  - A step is an elementary operation such as `+` or `<` or `A[j]`

Analysis of `Sum` (2)

```
int Sum(int A[], int N) {
  int sum = 0;           1

  for ( int j = 0; j < N; j++ )
                 2      3     4
    sum = sum + A[j];          5
7                        6
  return sum;       8
}
```

- 1, 2, 8: Once
- 3, 4, 5, 6, 7: Once per each iteration of `for`-loop
  - `N iterations`
- Total is `5N + 3` operations
- We can view this as a function of N, the complexity function of the algorithm: *f(N) = 5N + 3*.

How `5N+3` Grows

- The `5N+3` analysis gives an estimate of the true running time for different values of `N`:

  N = 10 => 53 steps

  N = 100 => 503 steps

  N = 1,000 => 5,003 steps

  N = 1,000,000 => 5,000,003 steps
- As `N` grows, the number of steps grows in *linear* proportion to `N`, for this `Sum` function

Methodology

- The example was typical
- 1. Analyze a program by counting steps
- 2. Derive a formula, based in some parameter N that is the size of the problem
  - For example, one algorithm might have a formula of N^2
  - Another might be 2^N
- 3. Study the formula to understand the overall efficiency

Why is this Useful?

What happens when we double the input size N?

N	$\log_2 N$	5N	N $\log_2 N$	N^2	2^N
8	3	40	24	64	256
16	4	80	64	256	65536
32	5	160	160	1024	$\sim 10^9$
64	6	320	384	4096	$\sim 10^{19}$
128	7	640	896	16384	$\sim 10^{38}$
256	8	1280	2048	65536	$\sim 10^{76}$
10000	**13**	**50000**	**10^5**	**10^8**	**$\sim 10^{3010}$**

Isn't This Totally Bogus?

- Need to run faster? Buy a faster computer!
  - Or just wait a while: CPU speed doubles every 18 months or so
  - "Moore's Law"
- Suppose we could make the CPU 1,000,000 times faster -- how much would that help?
  - Suppose the algorithm has complexity 2^N?
  - See following chart

If We Sped Up the CPU...

Even speeding up by a factor of a million, 10^{3010} **is only reduced to** 10^{3004}

N	$\log_2 N$	5N	$N \log_2 N$	N^2	2^N
8	3	40	24	64	256
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How long is a Computer-Day?

If my program needs f(N) microseconds to solve some problem, how big a problem can I solve in a day?

What if I get a million x faster computer ?

f(N)	N for 1 day	million x, N for 1 day
N	$N = 9 \times 10^{10}$	million times larger
5N	$N = 2 \times 10^{10}$	million times larger
$N \log_2 N$	$N = 3 \times 10^9$	60,000 times larger
N^2	$N = 3 \times 10^5$	1,000 times larger
N^3	$N = 4 \times 10^3$	100 times larger
2^N	$N = 36$	+20 larger

Big numbers

- Suppose a program has run time proportional to n!
- Suppose the run time for n = 10 is 1 second
- Do the math:
  - For n = 12, the run time is 2+ minutes
    `The time for 12 is 12! = 10! x 11 x 12 which is 132 times longer that 1 second: 132 seconds`
  - For n = 14, the run time is 6 hours
    `11 x 12 x 13 x 14 times longer`
  - For n = 16, the run time is 2 months
  - For n = 18, the run time is 50 years
  - For n = 20, the run time is 200 centuries

What Matters in the Long Run?

- What about the `5` in `5N+3`? What about the `+3`?
  - As `N` gets large, the `+3` becomes insignificant
  - The `5` is inaccurate:
    `<, [], +, =, ++ require varying amounts of time; different computers by and large differ by a constant factor`
- What is fundamental is that the time is *linear* in N
  - We say "5N+3 grows like N", or "5N+3 is *asymptotically* linear" or "5N+3 is asymptotically bounded by N", etc.

Asymptotic Complexity

- Asymptotic: what happens as N gets large
  - Focus on the highest-order term
    `Drop lower order terms such as +3`
  - Drop the constant coefficient of the highest order term
- This gives us an approximation of the complexity of the algorithm
  - Ignores lots of details, concentrates on the bigger picture

Comparing Algorithms

- We can now (partially) answer the question, “Given algorithms A and B, which is more efficient?”
  - Same as asking "Which algorithm has the smaller asymptotic time bound?"
- For specific values of `N`, we might get different (and uninformative) answers
- Instead, compare the growth *rates* for arbitrarily large values of `N` (the *asymptotic* case)

Comparing Functions

Definition: If f(N) and g(N) are two complexity functions, we say

`f(N) = O(g(N))`

(read "f(N) is order g(N)", or "f(N) is big-O of g(N)")

if there is a constant c such that

`f(N) ≤ c g(N)`

for all sufficiently large N.

Big-O Notation

- Think of **f(N) = O(g(N))** as
  " f(N) grows at most like g(N) " or
  "f grows no faster than g"
  (ignoring constant factors, and for large N)
- Big-O is not a function!
- Never read **=** as "equals"!
- Examples:
  - $5N + 3 = O(N)$
    also true: $5N + 3 = O(N^2)$
  - $37N^5 + 7N^2 - 2N + 1 = O(N^5)$

Computer Science Footnote

- There's a whole big theory of algorithmic complexity
- Typical questions:
  - What is the worst case performance (upper bound) of a particular algorithm?
  - What is the average case performance of a particular algorithm?
  - What is the best possible performance (lower bound) for a particular type of problem?
- Many difficult questions
  - Complicated mathematics
  - Still many unsolved problems!

"Computer Science is no more about computers than astronomy is about telescopes."

-- E. W. Dijkstra

Common Orders of Growth

Let N be the input size

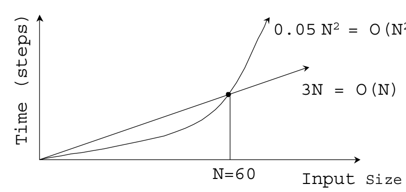
$O(k) = O(1)$	Constant Time	Increasing Complexity ↓
$O(\log_b N) = O(\log N)$	Logarithmic Time	
$O(N)$	Linear Time	
$O(N \log N)$		
$O(N^2)$	Quadratic Time	
$O(N^3)$	Cubic Time	
...		
$O(k^N)$	Exponential Time	

$N^{anyinteger}$ is called "polynomial" time

Rule of thumb: if it ain't polynomial, it ain't practical

Why is this Useful? (2)

- As inputs get larger, *any* algorithm of a smaller order will be more efficient than an algorithm of a larger order



Big-O Arithmetic

- Remember common functions in order from smallest to largest:
  `1, log(N), N, Nlog(N), N², N³,...,`
  `2^N, 3^N,...`
- Ignore constant multipliers
  $300\ N + 5N^4 + 6 \cdot 2^N = O(N + N^4 + 2^N)$
- Ignore everything except the highest order term
  $N + N^4 + 2^N = O(2^N)$

Constant Time Statements

Simplest case: `O(1)` time statements

- Assignment statements of simple data types
  `int x = y;`
- Arithmetic operations
  `x = 5 * y + 4 * z;`
- Array referencing
  `A[j]`
- Referencing/dereferencing pointers
  `Cursor = Head -> Next;`
- Declarations of simple data types
  `int x, y=3;`
- Most conditional tests
  `if ( x < 12 ) ...`

Constant Time Statements (2)

Watch out for things that look like simple `O(1)` time operations, but are actually more complex:

- Overloaded operators
```
LinkedList L1 (L2); // deep copy?
myList s1 = s2 + s3;   // overloaded + ??
```
- Declaring complex data types that have constructors
- Function invocations
```
if ( aPriorityQueue.Size() < 10 ) ...
```


Analyzing Loops

Any loop analysis has two parts:

1. How many iterations are performed?
2. How many steps per iteration?

```
int sum = 0;
for (int j = 0; j < N; j++ )
  sum = sum + j ;
```

- Loop executes `N` times (`0 .. N-1`)
- `4 = O(1)` steps per iteration
- Total time is $N \cdot O(1) = O(N \cdot 1) = O(N)$

Analyzing Loops (2)

- What about this `for`-loop?
```
int sum = 0;
for (int j = 0; j < 100; j++ )
  sum = sum + j;
```
  - Loop executes `100` times (`0 .. 99`)
  - `4 = O(1)` steps per iteration
  - Total time is $100 \cdot O(1) = O(100 \cdot 1) = O(100) = \mathbf{O(1)}$
- That this loop is faster makes sense when `N >> 100`.

Analyzing Loops (3)

What about `while`-loops?

- Determine how many times the loop will be executed
```
bool done = false;
int result = 1, n;
cin >> n;
while ( !done ) {
  result = result * n;
  n--;
  if ( n <= 1 ) done = true;
}
```
- Loop terminates when `done == true`, which happens after `n` iterations
- `O(1)` time per iteration
- `O(n)` total time

Nested Loops –Easy Case

- Treat just like a single loop, and evaluate each level of nesting as needed:

```
int j, k, sum = 0;
for ( j = 0; j < N; j++ )
   for ( k = N; k > 0; k-- )
      sum += k + j;
```

- Start with outer loop:
  - How many iterations? `N`
  - How much time per iteration? Need to evaluate inner loop …
- Inner loop uses `O(N)` time
  - and this does not depend on the outer loop time
- Total is $N \cdot O(N) = O(N \cdot N) = O(N^2)$

Nested Loops – Harder Case

- What if the number of iterations of one loop depends on the counter of the other?

```
int j, k, sum = 0;
for ( j = 0; j < N; j++ )
   for ( k = 0; k < j; k++ )
      sum += k * j;
```

- Analyze inner and outer loops together
- For this example, number of iterations of the inner loop is

$0 + 1 + 2 + \ldots + (N-1) = O(N^2)$

  - Time per iteration is $O(1)$, for total $O(N^2)$
- In general, finding a formula can be hard

Sequences of Statements

For a sequence of statements, compute their cost functions individually and add them up

```
for (int j = 0; j < N; j++)
   for (int k = 0; k < j; k++)
      sum = sum + j*k;                       } O(N^2)
for (int l = 0; l < N; l++ )
   sum = sum - l;                            } O(N)
cout << "Sum is now " << sum << endl;       } O(1)
```

Total cost is $O(N^2) + O(N) + O(1) = O(N^2)$

Conditional Statements

- What about a conditional statement such as

```
if (condition)
    statement1;
else
    statement2;
```

where `statement1` runs in O(n) time and `statement2` runs in O(n^2) time?
- We use "worst-case complexity": among all inputs of size n, what is the *maximum* running time?
- The analysis for the example above is O(n^2).

"Worst-Case" vs "Average-Case"

```
if (condition)
    statement1;
else
    statement2;
```

- If you knew how often the condition is true, you could compute a weighted average.
  - Extreme case: the conditional might be always true or never true
- "Average case" analysis can be very difficult
  - Use tools from probability and statistics
- For many algorithms, it is useful to know both the worst case and the average case complexity

Cost of Function Calls

F (b, c);

- Cost =

```
cost of making the call
+ cost of passing the arguments
+ cost of executing the function
```

- Making and returning from the call: O(1)
- Passing the arguments: depends on how they are passed
- Cost of execution: must do analysis of the function itself

Efficiency in Parameter Passing

- Pass by value -- copies entire structure
  - Page::Translate(CodeBook cb);
  - What if there's a copy constructor?
- Pass by reference -- does not copy, but allows updates
  - Page::Translate(CodeBook& cb);
  - Page::Translate(CodeBook * cb);
- const reference -- pass by reference, but do not allow changes
  - Page::Translate(**const** CodeBook& cb);
- *Which technique should you use??*

Recursive Algorithms

- We need to know two things:
  - number of recursive calls
  - the work done at each level of recursion
- Example: exponentiation

```
int exp (int x, int n) {
   if (n==0)
     return 1;          } O(1)
   else
     return x * exp(x,n-1);
}
```

  - The running time is O(n):

    n recursive calls until base case is reached, and the work done at each call is O(1)
- In general, a "recurrence relation" results from the analysis, solvable with tools from math.

Recursive Algorithms (2)

- Fibonacci numbers:

```
int fib (int n) {
  if (n == 1 || n == 2)
    return 1;
  else
    return fib(n-1) + fib(n-2);
}
```

  - How many calls? How much work at each call?
  - Recurrence relation: T(n) = T(n-1)+T(n-2)+O(1)
  - Running time? Solve the equation

List Implementations

N is the list size

	array	linked list	doubly linked list
constructor			
isEmpty			
isFull			
reset			
advance			
endOfList			
data			
size			
insertBefore			
insertAfter			
deleteItem			

List Implementations

N is the list size

	array	linked list	doubly linked list
constructor	O(1)	O(1)	O(1)
isEmpty	O(1)	O(1)	O(1)
isFull	O(1)	O(1)	O(1)
reset	O(1)	O(1)	O(1)
advance	O(1)	O(1)	O(1)
endOfList	O(1)	O(1)	O(1)
data	O(1)	O(1)	O(1)
size	O(1)	O(N)	O(N)
insertBefore	O(N)	O(N)	O(1)
insertAfter	O(N)	O(1)	O(1)
deleteItem	O(N)	O(N)	O(1)

Dynamic arrays

- When array is full, reallocate new array
  - increase by one

```
if (size == maxSize){
  int *tmp = new int[maxSize + 1];
  …
}
```

  - double array

```
if (size == maxSize){
  int *tmp = new int[2*maxSize];
  …
}
```

- What is the cost of

```
Vector a;
for (int i = 0; i < N; i++)
    a.insert(i, i);
```

Dynamic Array Analysis

- Count the sizes of the arrays allocated
- Increment by one:
  - $1 + 2 + 3 + 4 + \ldots + n = O(N^2)$
- Double size (assume n is a power of 2)
  - $1 + 2 + 4 + 8 + 16 + \ldots + n/4 + n/2 + n = 2N - 1 = O(N)$

8/7/00 U-43

Printing a list in reverse order

- Iterative

```
L.GoToEndOfList();
while (! L.StartOfList()){
  L.Previous();
  cout << L.Data();
}
```

  - $O(N^2)$ since Previous is O(N)
- Recursive

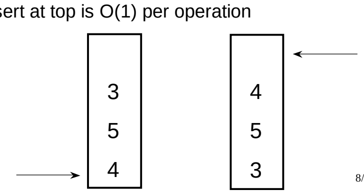
```
void List::RevPrint(){
  if (EndOfList()) return;
  int d = Data(); Advance();
  RevPrint(); cout << d;
}
```

  - O(N), N recursive calls at O(1) each

8/7/00 U-44

Stack implementation

- Measure cost of inserting N elements
- Array implementation
  - Insert at bottom (dumb) vs. insert at top
  - Insert at bottom: O(N) to insert element since everything is copied
  - Insert at top is O(1) per operation



8/7/00 U-45

Review: Common Orders of Growth

Memorize!

`O(k) = O(1)`	Constant Time
$O(\log_b N) = O(\log N)$	Logarithmic Time
`O(N)`	Linear Time
`O(N log N)`	
$O(N^2)$	Quadratic Time
$O(N^3)$	Cubic Time
...	
$O(k^N)$	Exponential Time

Increasing Complexity

$N^{anyinteger}$ is called "polynomial" time

Rule of thumb: if it ain't polynomial, it ain't practical

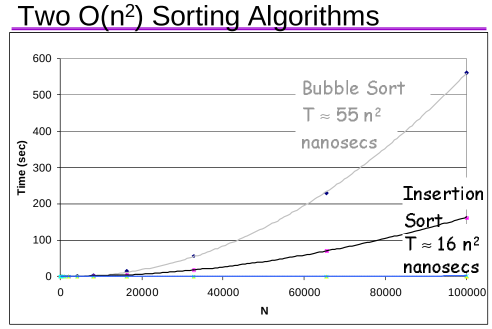
8/7/00 U-46

Summary

- Measuring Efficiency
  - Measure as a function of input size N
  - Use steps of time or units of memory for measurements
- Asymptotic complexity
  - Growth rate as N gets large
- Order of common complexity functions
- Big-O notation
- Methods for analyzing programs

8/7/00 U-47

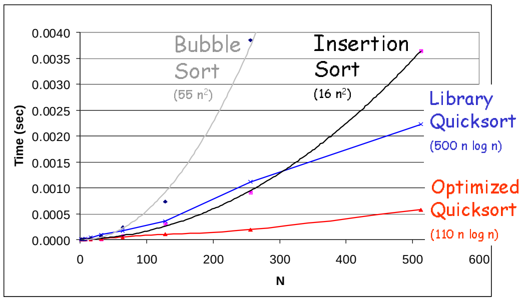
Do Constants Matter? Two $O(n^2)$ Sorting Algorithms



Benchmark run on 233Mhz P II, 96M, NT 4.0, VC 6.0, 5/30/00

8/7/00 U-48

O(n log n) vs O(n^2): closeup



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Summary

- FIRST pick the right algorithm
  - Big-O helps do that
  - Can give *many* orders of magnitude improvement
- THEN optimize it
  - above 2x improvement is uncommon

Premature optimization is the root of all evil -- D. Knuth

8/7/00 U-50