Affine transformations

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Adapted from Brian Curless
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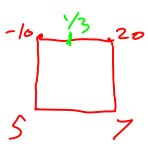
Reading

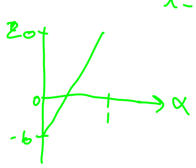
Optional reading:

- Angel 4.1, 4.6-4.10
- Angel, the rest of Chapter 4
- Foley, et al, Chapter 5.1-5.5.
- David F. Rogers and J. Alan Adams, *Mathematical Elements for Computer Graphics*, 2nd Ed., McGraw-Hill, New York, 1990, Chapter 2.

Linear Interpolation



$l = -10 + 30\alpha$



$0 = -10 + 30\alpha$

$10 = 30\alpha$

$\frac{1}{3} = \alpha$

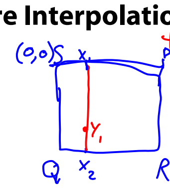
$-10(1-\alpha) + 20\alpha$

$0 \le \alpha \le 1$

B F

$I = B(1-\alpha) + F\alpha$

More Interpolation



$f(0.2, 0.7) = y_1$

$x_1 = S(1-0.2) + P(0.2)$

$x_2 = Q(1-0.2) + R(0.2)$

$y_1 = x_1(1-0.7) + x_2(0.7)$

Geometric transformations

Geometric transformations will map points in one space to points in another: $(x', y', z') = \boldsymbol{f}(x, y, z)$.

These transformations can be very simple, such as scaling each coordinate, or complex, such as non-linear twists and bends.

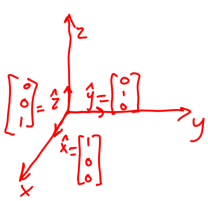
We'll focus on transformations that can be represented easily with matrix operations.

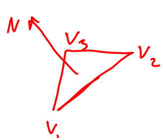
Vector representation

We can represent a **point**, **p** = (x,y), in the plane or p=(x,y,z) in 3D space

- as column vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ ✓
- as row vectors $\begin{bmatrix} x & y \end{bmatrix}$ $\begin{bmatrix} x & y & z \end{bmatrix}$

Canonical axes





Vector length and dot products

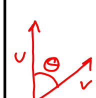
$U = \begin{bmatrix} U_x \\ U_y \\ U_z \end{bmatrix}$ $\|U\| = \sqrt{U_x^2 + U_y^2 + U_z^2}$

$\hat{U} = \frac{U}{\|U\|}$ Unit vector, Direction vector

$V = \begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix}$ $\rightarrow U \cdot V = U_x V_x + U_y V_y + U_z V_z$

$U \cdot V = U^T V = \begin{bmatrix} U_x & U_y & U_z \end{bmatrix} \begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix}$

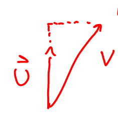
$U \cdot V \stackrel{?}{=} V \cdot U$ true



$U \cdot V = \|U\| \|V\| \cos\theta$

$U \cdot V = 0$ $\begin{cases} \theta = 90^\circ \text{ or } 270^\circ \perp \text{ orthogonal} \\ \|U\| = 0 \\ \|V\| = 0 \end{cases}$

$\|U\| = \|V\| = 1 \Rightarrow U \cdot V = \cos\theta$



$\hat{U} \cdot V = \|V\| \cos\theta$

$U \cdot U = \|U\|^2$

Vector cross products

$$u \times v = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} = \hat{x}(u_y v_z) - (v_y u_z) - [\hat{y}(u_x v_z) - (v_x u_z)] + \hat{z}(u_x v_y) - (v_x u_y)$$

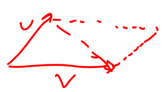
$u \times v \stackrel{?}{=} v \times u$ false

$u \times v = -v \times u$

$(u \times v) \cdot u = 0$

$\|u \times v\| = \|u\| \, \|v\| \sin\theta$

$u \times u = 0$



$\|u \times v\| =$ area ▱

Inverse & Transpose

$A A^{-1} = I$

$(AB)^{-1}(AB) = I$

$(AB)^{-1} A B B^{-1} = I B^{-1}$

$(AB)^{-1} A A^{-1} = B^{-1} A^{-1}$

$(AB)^{-1} = B^{-1} A^{-1}$

$(AB)^T = B^T A^T$

Representation, cont.

We can represent a **2-D transformation** *M* by a matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

If **p** is a column vector, *M* goes on the left:

$$\mathbf{p}' = M\mathbf{p}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

If **p** is a row vector, M^T goes on the right:

$$\mathbf{p}' = \mathbf{p} M^T$$

$$\begin{bmatrix} x' & y' \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

We will use **column vectors**.

Two-dimensional transformations

Here's all you get with a 2 x 2 transformation matrix *M*:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

So:

$$x' = ax + by$$
$$y' = cx + dy$$

We will develop some intimacy with the elements *a, b, c, d…*

Identity

Suppose we choose $a=d=1$, $b=c=0$:

- Gives the **identity** matrix:

$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

- Doesn't move the points at all

13

Scaling

Suppose we set $b=c=0$, but let a and d take on any *positive* value:

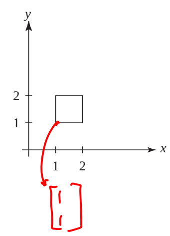
- Gives a **scaling** matrix:

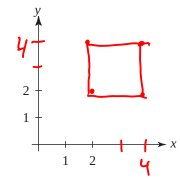
$$\begin{bmatrix}a & 0\\0 & d\end{bmatrix}$$

- Provides **differential (non-uniform) scaling** in x and y:

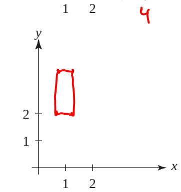
$$x' = ax \qquad y' = dy$$

$a=d \Rightarrow$ uniform Scaling





$$\begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}2x\\2y\end{bmatrix}$$



$$\begin{bmatrix}1/2 & 0\\0 & 2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x/2\\2y\end{bmatrix}$$

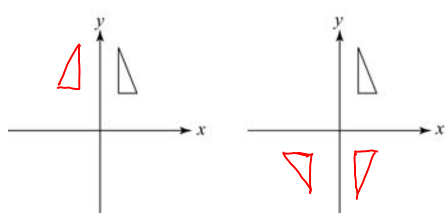
14

Reflection

Suppose we keep $b=c=0$, but let either a or d go negative.

Examples:

$$\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}-x\\y\end{bmatrix} \qquad \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x\\-y\end{bmatrix}$$



$$\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}-x\\-y\end{bmatrix}$$

15

Skew Shear

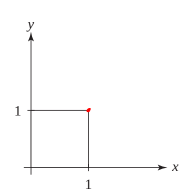
Now let's leave $a=d=1$ and experiment with b. . . .

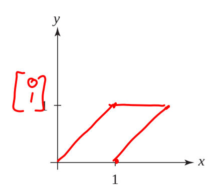
The matrix

$$\begin{bmatrix}1 & b\\0 & 1\end{bmatrix} \qquad \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$$

gives:

$$x' = x + by \qquad y' = y$$





$$\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$$

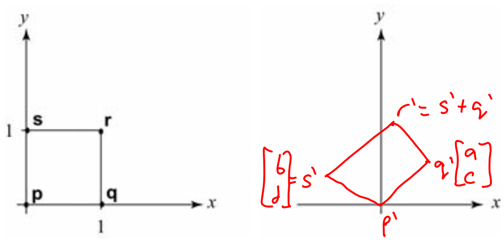
16

Effect on unit square

Let's see how a general 2 x 2 transformation *M* affects the unit square:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} [\mathbf{p} \quad \mathbf{q} \quad \mathbf{r} \quad \mathbf{s}] = [\mathbf{p'} \quad \mathbf{q'} \quad \mathbf{r'} \quad \mathbf{s'}]$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & a & a+b & b \\ 0 & c & c+d & d \end{bmatrix}$$



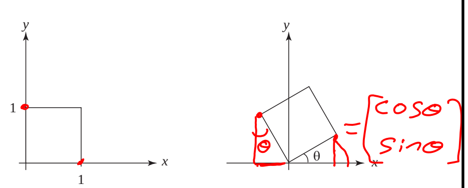
Effect on unit square, cont.

Observe:

- Origin invariant under *M*
- *M* can be determined just by knowing how the corners (1,0) and (0,1) are mapped
- *a* and *d* give *x*- and *y*-scaling
- *b* and *c* give *x*- and *y*-shearing

Rotation

From our observations of the effect on the unit square, it should be easy to write down a matrix for "rotation about the origin":



- $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$
- $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$

Thus,

$$M = R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Limitations of the 2 x 2 matrix

A 2 x 2 linear transformation matrix allows

- Scaling
- Rotation
- Reflection
- Shearing

Q: What important operation does that leave out?

Translate

Homogeneous coordinates

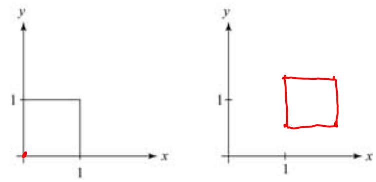
We can loft the problem up into 3-space, adding a third component to every point:

$$\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Adding the third "*w*" component puts us in **homogenous coordinates**.

Then, transform with a 3 x 3 matrix:

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = T(\mathbf{t}) \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{array}{c} \quad A \qquad t \\ \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \end{array} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$



$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \\ 1 \end{bmatrix}$$

. . . gives **translation**!

Affine transformations

The addition of translation to linear transformations gives us **affine transformations**.

In matrix form, 2D affine transformations always look like this:

$$M = \begin{bmatrix} a & b & t_x \\ c & d & t_y \\ 0 & 0 & 1 \end{bmatrix} = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{A} & \mathbf{t} \\ \hline 0 & 0 & 1 \end{array}\right]$$

2D affine transformations always have a bottom row of [0 0 1].

An "affine point" is a "linear point" with an added *w*-coordinate which is always 1:

$$\mathbf{p}_{\text{aff}} = \begin{bmatrix} \mathbf{p}_{\text{lin}} \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

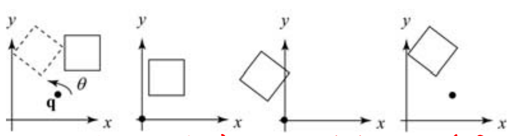
Applying an affine transformation gives another affine point:

$$M\mathbf{p}_{\text{aff}} = \begin{bmatrix} A\mathbf{p}_{\text{lin}} + \mathbf{t} \\ 1 \end{bmatrix}$$

Rotation about arbitrary points

Until now, we have only considered rotation about the origin.

With homogeneous coordinates, you can specify a rotation, θ, about any point $\mathbf{q} = [q_x \; q_y \; 1]^T$ with a matrix:



$M \neq T(-q) \cdot R(\theta) \cdot T(q)$

$M = T(q) R(\theta) T(-q)$

1. Translate **q** to origin
2. Rotate
3. Translate back

$$T(v) = \begin{bmatrix} 1 & 0 & v_x \\ 0 & 1 & v_y \\ 0 & 0 & 1 \end{bmatrix}$$

$$R(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

<u>Note</u>: Transformation order is important!!

Points and vectors

Vectors have an additional coordinate of *w*=0. Thus, a change of origin has no effect on vectors.

Q: What happens if we multiply a vector by an affine matrix?

These representations reflect some of the rules of affine operations on points and vectors:

vector + vector	$\rightarrow$	vector
scalar · vector	$\rightarrow$	vector
point - point	$\rightarrow$	vector
point + vector	$\rightarrow$	point
point + point	$\rightarrow$	chaos

One useful combination of affine operations is:

$$\mathbf{p}(t) = \mathbf{p}_o + t\mathbf{u}$$

Q: What does this describe?

line $\quad 0 \le t \Rightarrow$ half line or ray

$p \rightarrow q$

$$q - p = \begin{bmatrix} q_x \\ q_y \\ 1 \end{bmatrix} - \begin{bmatrix} p_x \\ p_y \\ 1 \end{bmatrix} = \begin{bmatrix} q_x - p_x \\ q_y - p_y \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} A & \vec{t} \\ 0\,0 & 1 \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ 0 \end{bmatrix} = \begin{bmatrix} \xi \\ 0 \end{bmatrix}$$

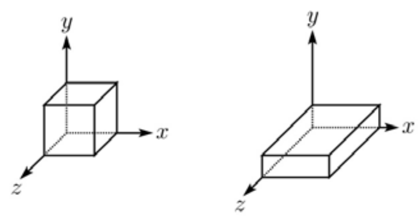
Basic 3-D transformations: scaling

Some of the 3-D affine transformations are just like the 2-D ones.

In this case, the bottom row is always [0 0 0 1].

For example, scaling: $S(s_x, s_y, s_z)$ $\quad S^{-1}\left(\frac{1}{s_x}, \frac{1}{s_y}, \frac{1}{s_z}\right)$

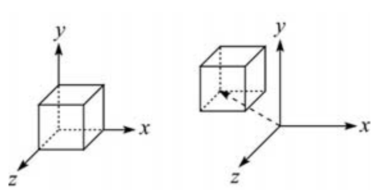
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



Translation in 3D

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$T^{-1}(\vec{t}) = T(-\vec{t})$$



Rotation in 3D

$R[u\ v\ w]$

$$R^T = \begin{bmatrix} u^T \\ v^T \\ w^T \end{bmatrix}$$

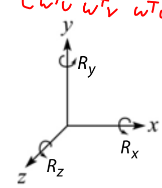
Rotation now has more possibilities in 3D:

$$R_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R^T R = \begin{bmatrix} u^T u & u^T v & u^T w \\ v^T u & v^T v & v^T w \\ w^T u & w^T v & w^T w \end{bmatrix}$$

$$R_y(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\beta & 0 & \cos\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_z(\gamma) = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 & 0 \\ \sin\gamma & \cos\gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Use right hand rule

$$R^T R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

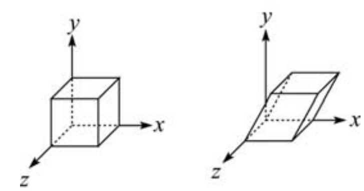
A general rotation can be specified in terms of a prodcut of these three matrices. How else might you specify a rotation?

quaternions

Shearing in 3D

Shearing is also more complicated. Here is one example:

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & b & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



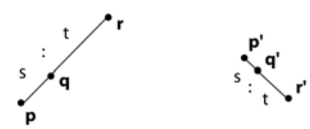
We call this a shear with respect to the x-z plane.

29

Properties of affine transformations

Here are some useful properties of affine transformations:

- Lines map to lines
- Parallel lines remain parallel
- Midpoints map to midpoints (in fact, ratios are always preserved)



$$\text{ratio} = \frac{\|\mathbf{pq}\|}{\|\mathbf{qr}\|} = \frac{s}{t} = \frac{\|\mathbf{p'q'}\|}{\|\mathbf{q'r'}\|}$$

30

Affine transformations in OpenGL

OpenGL maintains a "modelview" matrix that holds the current transformation **M.**

The modelview matrix is applied to points (usually vertices of polygons) before drawing.

It is modified by commands including:

- `glLoadIdentity()` **M ← I**
  - set **M** to identity
- `glTranslatef(`t_x`,` t_y`,` t_z`)` **M ← MT**
  - translate by (t_x, t_y, t_z)
- `glRotatef(θ, x, y, z)` **M ← MR**
  - rotate by angle θ about axis (x, y, z)
- `glScalef(`s_x`,` s_y`,` s_z`)` **M ← MS**
  - scale by (s_x, s_y, s_z)

Note that OpenGL adds transformations by *postmultiplication* of the modelview matrix.

31

Summary

What to take away from this lecture:

- All the names in boldface.
- How points and transformations are represented.
- How to compute lengths, dot products, and cross products of vectors, and what their geometrical meanings are.
- What all the elements of a 2 x 2 transformation matrix do and how these generalize to 3 x 3 transformations.
- What homogeneous coordinates are and how they work for affine transformations.
- How to concatenate transformations.
- The mathematical properties of affine transformations.

32