Practical Aspects of Modern Cryptography

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Public-Key History

- 1976 *New Directions in Cryptography*

  Whit Diffie and Marty Hellman

  - One-Way functions
  - Diffie-Hellman Key Exchange

- 1978 RSA paper

  Ron Rivest, Adi Shamir, and Len Adleman

  - RSA Encryption System
  - RSA Digital Signature Mechanism

The Fundamental Equation

$$Z \equiv Y^X \bmod N$$

Diffie-Hellman

$$Z \equiv Y^X \bmod N$$

When X is unknown, the problem is known as the *discrete logarithm* and is generally believed to be hard to solve.

Diffie-Hellman Key Exchange

Alice

- Randomly select a large integer a and send $A = Y^a \bmod N$.
- Compute the key $K = B^a \bmod N$.

Bob

- Randomly select a large integer b and send $B = Y^b \bmod N$.
- Compute the key $K = A^b \bmod N$.

$$B^a = Y^{ba} = Y^{ab} = A^b$$

One-Way Trap-Door Functions

$$Z \equiv Y^X \bmod N$$

Recall that this equation is solvable for Y if the factorization of N is known, but is *believed* to be hard otherwise.

RSA Public-Key Cryptosystem

Alice

- Select two large random primes P & Q.
- Publish the product $N=PQ$.

- Use knowledge of P & Q to compute Y.

Anyone

- To send message Y to Alice, compute $Z=Y^X \bmod N$.
- Send Z and X to Alice.

Some RSA Details

When $N=PQ$ is the product of distinct primes,

$$Y^X \bmod N = Y$$

whenever

$X \bmod (P-1)(Q-1) = 1$ and $0 \leq Y < N$.

Alice can easily select integers E and D such that $E \bullet D \bmod (P-1)(Q-1) = 1$.

Remaining RSA Basics

- Why is $Y^X \bmod PQ = Y$ whenever

  $X \bmod (P-1)(Q-1) = 1$, $0 \leq Y < PQ$,

  and P and Q are distinct primes?

- How can Alice can select integers E and D such that $E \bullet D \bmod (P-1)(Q-1) = 1$?

Fermat's Little Theorem

If p is prime,

then $x^{p-1} \bmod p = 1$ for all $0 < x < p$.

Equivalently …

If p is prime,

then $x^{p} \bmod p = x \bmod p$ for all integers x.

Proof of Fermat's Little Theorem

The Binomial Theorem

$$(x+y)^p = x^p + \binom{p}{1}x^{p-1}y + \ldots + \binom{p}{p-1}xy^{p-1} + y^p$$

where $\binom{p}{i} = \frac{p!}{i!(p-i)!}$

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where $\binom{p}{i} = \frac{p!}{i!(p-i)!}$

If p is prime, then $\binom{p}{i} \bmod p = 0$ for $0 < i < p$.

Thus, $(x+y)^p \bmod p = (x^p + y^p) \bmod p$.

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By induction on x…

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If $x = 0$, then $x^p \bmod p = 0 = x \bmod p$.

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If $x = 0$, then $x^p \bmod p = 0 = x \bmod p$.

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$= (x + 1) \bmod p$.

Proof of Fermat's Little Theorem

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Assume that $x^p \bmod p = x \bmod p$.

Then $(x + 1)^p \bmod p = (x^p + 1^p) \bmod p$

$= (x + 1) \bmod p$.

Hence, $x^p \bmod p = x \bmod p$ for integers $x \geq 0$.

Proof of Fermat's Little Theorem

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Assume that $x^p \bmod p = x \bmod p$.

Then $(x + 1)^p \bmod p = (x^p + 1^p) \bmod p$

$= (x + 1) \bmod p$.

Hence, $x^p \bmod p = x \bmod p$ for integers $x \geq 0$.

Also true for negative x, since $(-x)^p = (-1)^p x^p$.

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$$Y^{K(P-1)(Q-1)+1} \bmod PQ = Y \text{ when } 0 \leq Y < PQ$$

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$Y^P \bmod P = Y$ whenever $0 \leq Y < P$

and P is *prime*!

You will show …

$Y^{K(P-1)(Q-1)+1} \bmod PQ = Y$ when $0 \leq Y < PQ$

P and Q are distinct primes and $K \geq 0$.

Finding Primes

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- Suppose that the set of all primes were finite.
- Let N be the product of all of the primes.
- Consider N+1.
- The prime factors of N+1 are not among the finite set of primes multiplied to form N.
- This contradicts the assumption that the set of all primes is finite.

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Thus, approximately 1 out of every *n* randomly selected *n*-bit integers will be prime.

Testing Primality

Recall Fermat's Little Theorem

If p is prime, then $a^{(p-1)} \bmod p = 1$ for all a in the range $0 < a < p$.

The Miller-Rabin Primality Test

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- If $a^m = \pm 1$ or if some $a^{2^i m} = -1$, then N is probably prime – continue.
- Otherwise, N is composite – stop.

Sieving for Primes

Pick a random starting point N.

N	N+1	N+2	N+3	N+4	N+5	N+6	N+7	N+8	N+9	N+10	N+11

Sieving out multiples of 2

Sieving for Primes

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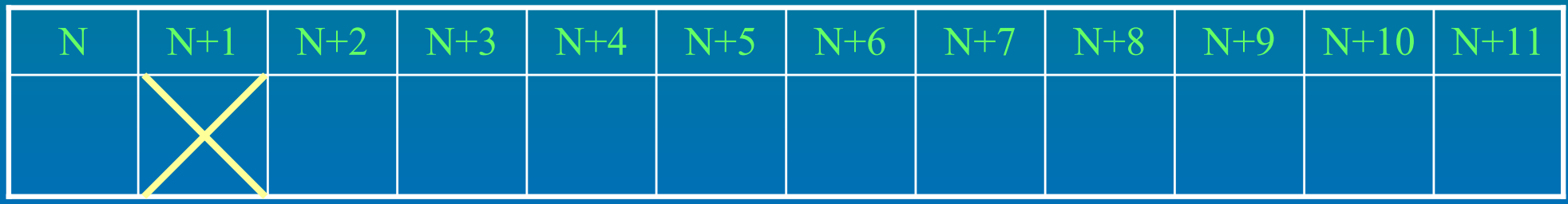

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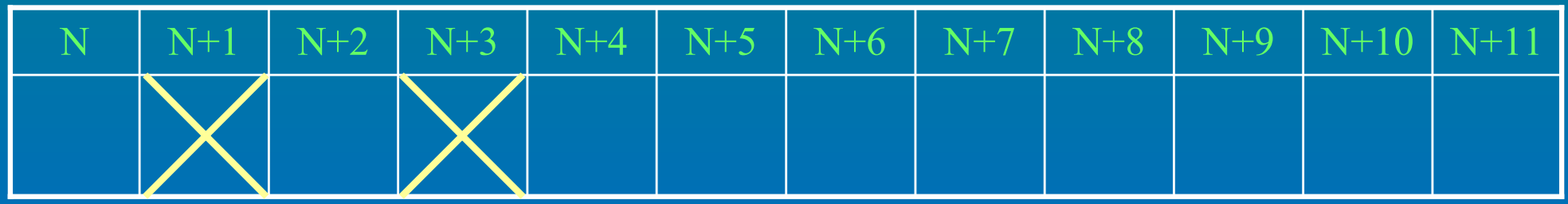

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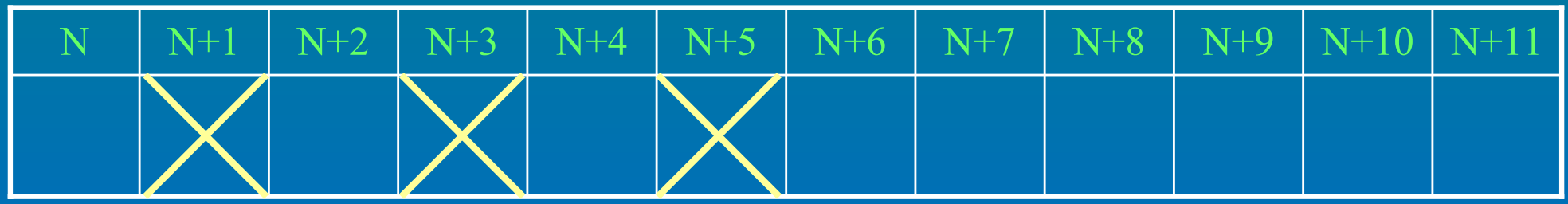

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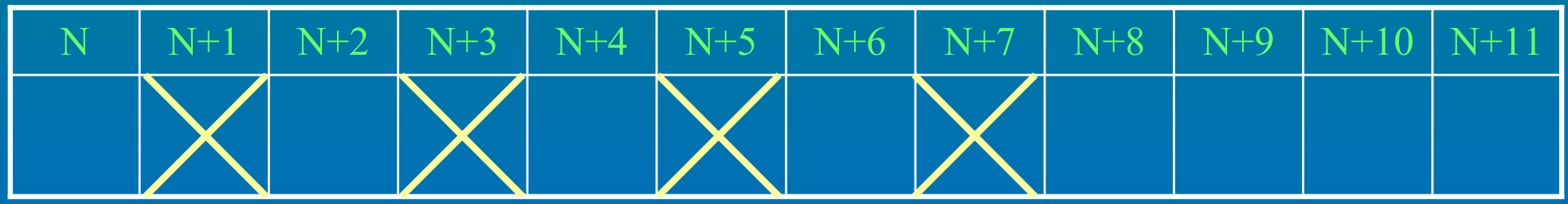

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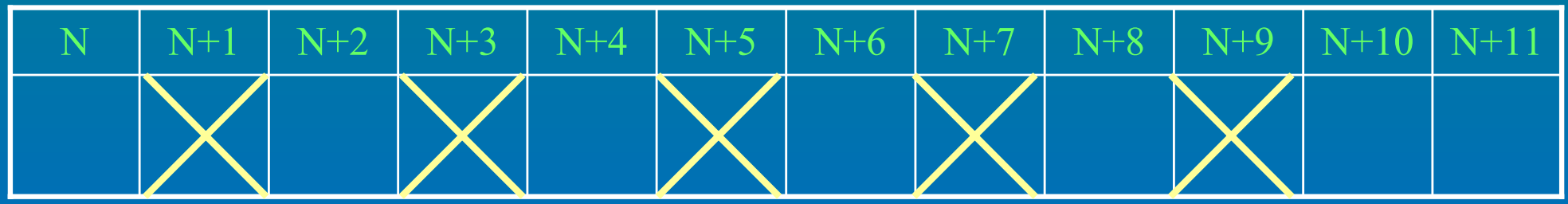

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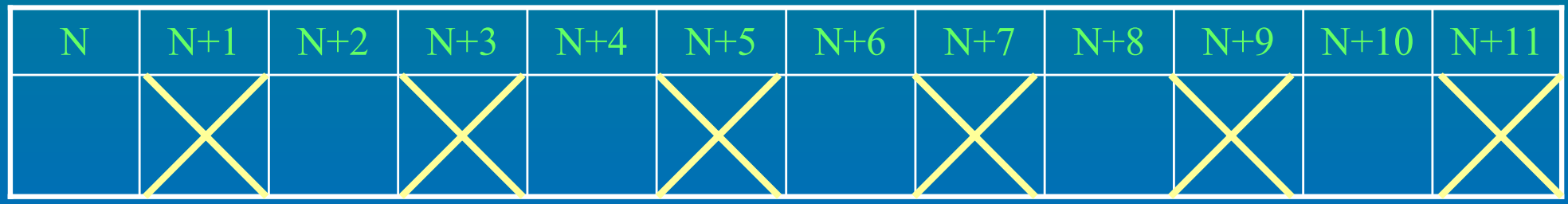

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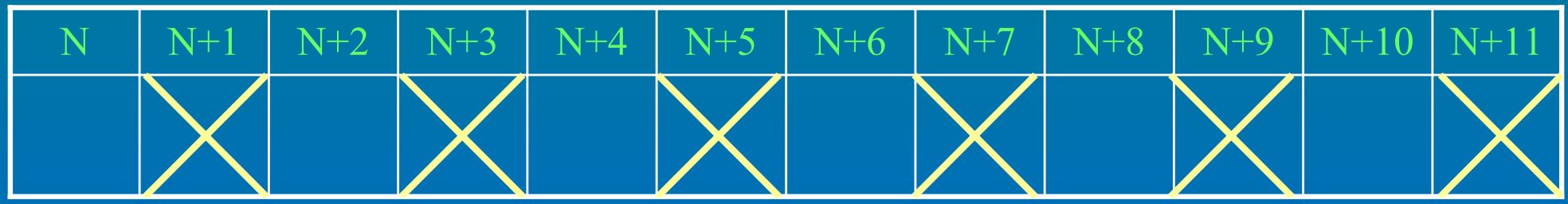
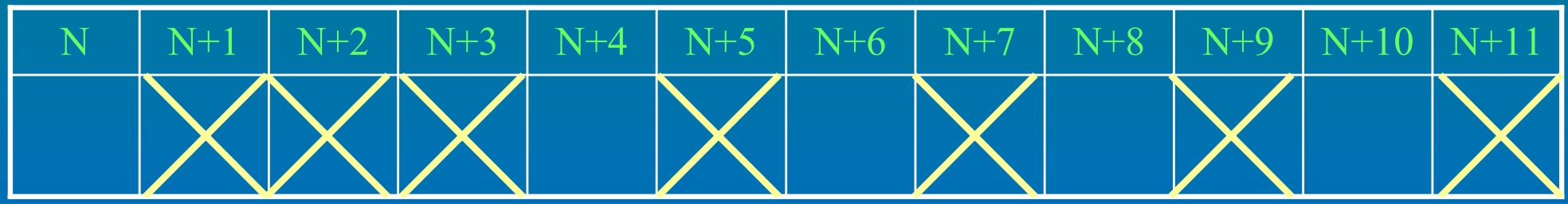


Sieving out multiples of 3

Sieving for Primes

Pick a random starting point N.



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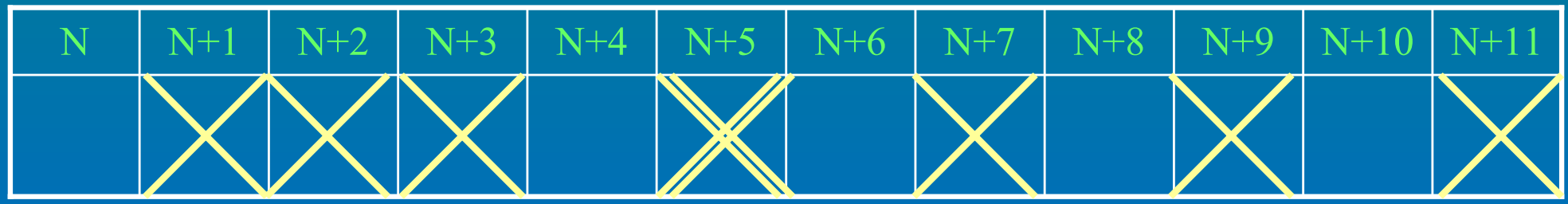

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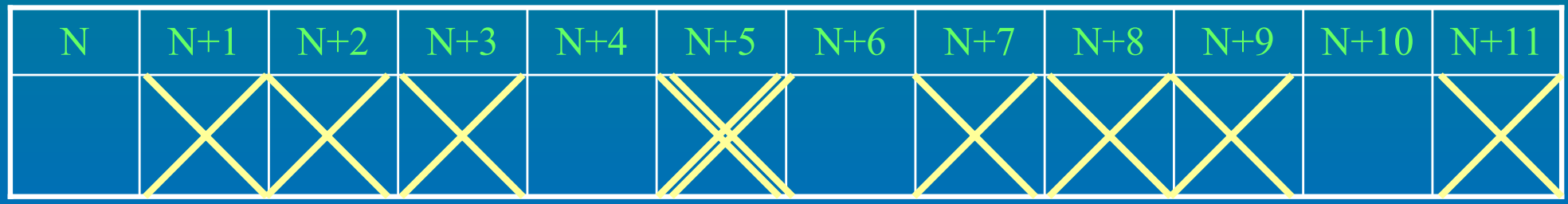

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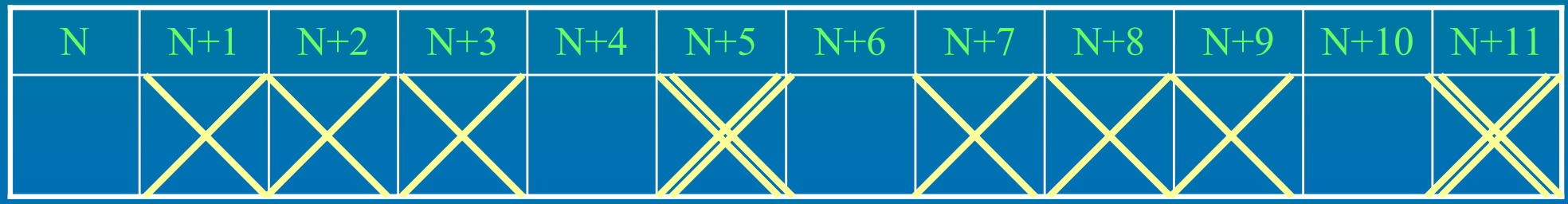

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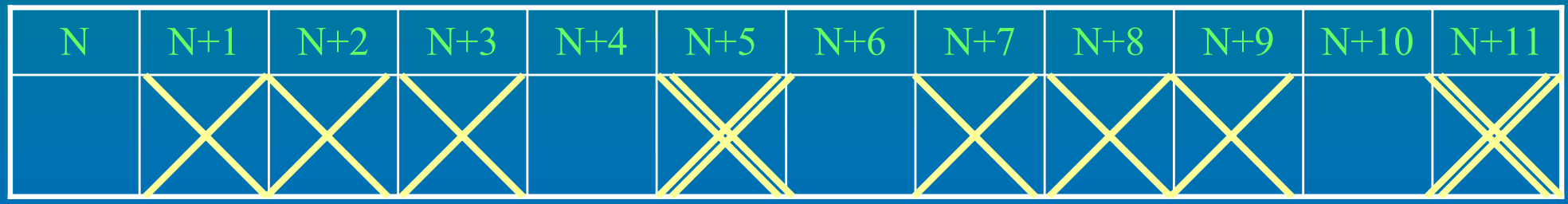
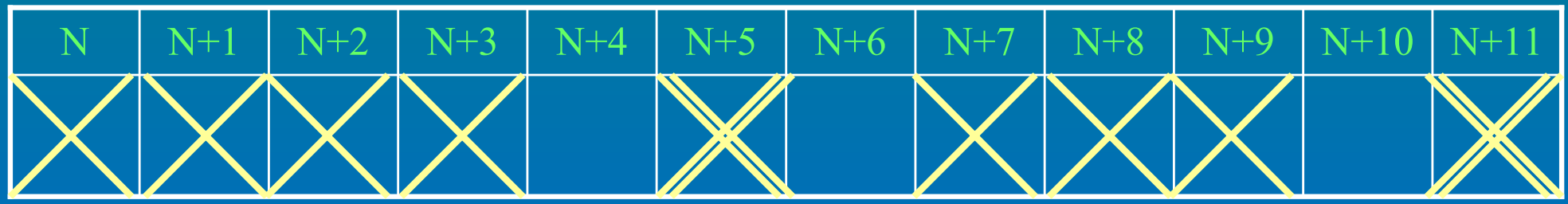


Sieving out multiples of 5

Sieving for Primes

Pick a random starting point N.



Sieving out multiples of 5

Sieving for Primes

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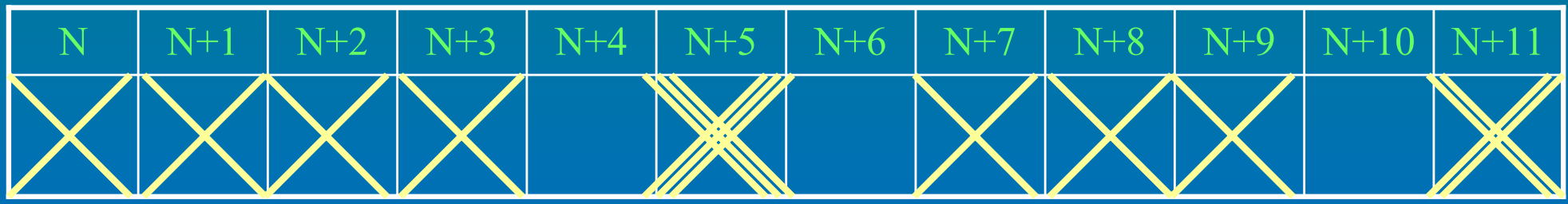

Sieving out multiples of 5

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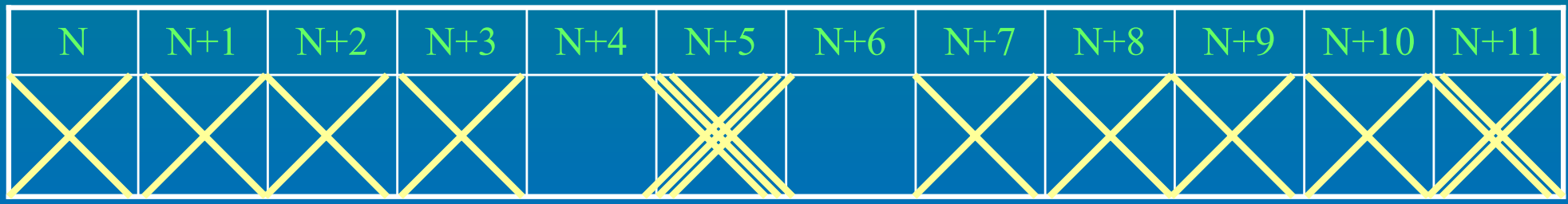

Sieving out multiples of 5

Sieving for Primes

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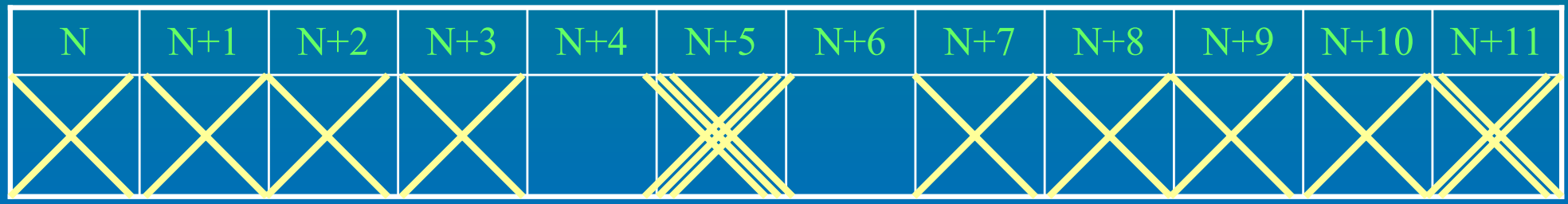

Sieving out multiples of 5

Only a few “good” candidate primes will survive.

Remaining RSA Basics

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- How can Alice can select integers E and D such that $E \bullet D \bmod (P-1)(Q-1) = 1$?

Modular Arithmetic

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- To compute $(A \div B) \bmod N$, …

Modular Division

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What is the value of $(1 \div 2) \bmod 7$?

We need a solution to $2x \bmod 7 = 1$.

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Try $x = 4$.

What is the value of $(7 \div 5) \bmod 11$?

We need a solution to $5x \bmod 11 = 7$.

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We need a solution to $2x \bmod 7 = 1$.

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What is the value of $(7 \div 5) \bmod 11$?

We need a solution to $5x \bmod 11 = 7$.

Try $x = 8$.

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Fact

$(A \div B) \bmod N$ always has a solution when $\gcd(B,N) = 1$.

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$(A \div B) \bmod N$ always has a solution when $\gcd(B,N) = 1$.*

*There is no solution if $\gcd(A,B) = 1$ and $\gcd(B,N) \neq 1$.

Greatest Common Divisors

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gcd(21,12) = gcd(12,9) = gcd(9,3)

= gcd(6,3) = gcd(3,6) = gcd(3,3)

= gcd(3,0) = 3

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gcd(A , B) = gcd(B , A – *k*B) for any integer *k*.

gcd(A , B) = gcd(B , A mod B)

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Extended Euclidean Algorithm

Given integers A and B, find integers X and Y such that $AX + BY = \gcd(A,B)$.

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Compute $(C \div A) \bmod B$ as $C \times (1 \div A) \bmod B$.

Extended Euclidean Algorithm

gcd(35, 8) =

gcd(8, 35 mod 8) = gcd(8, 3) =

gcd(3, 8 mod 3) = gcd(3, 2) =

gcd(2, 3 mod 2) = gcd(2, 1) =

gcd(1, 2 mod 1) = gcd(1, 0) = 1

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$$35 = 8 \times 4 + 3$$

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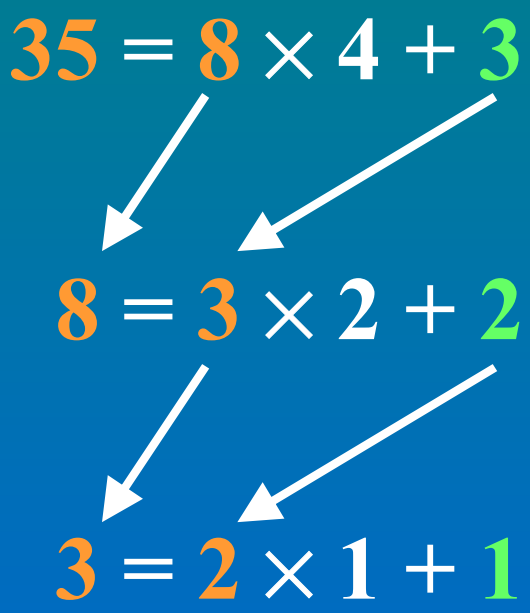
Extended Euclidean Algorithm



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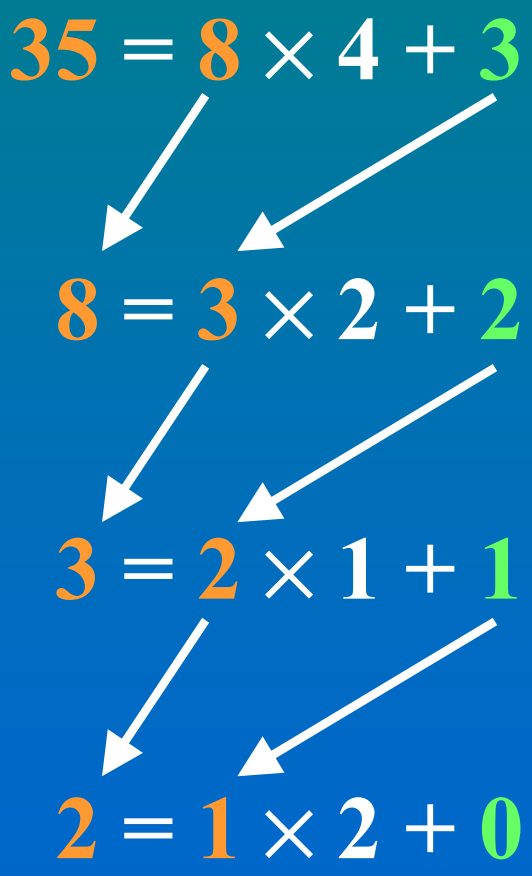

Extended Euclidean Algorithm

$35 = 8 \times 4 + 3$

$8 = 3 \times 2 + 2$

$3 = 2 \times 1 + 1$

$2 = 1 \times 2 + 0$

$3 = 35 - 8 \times 4$

$2 = 8 - 3 \times 2$

$1 = 3 - 2 \times 1$

Extended Euclidean Algorithm

$$3 = 35 - 8 \times 4$$

$$2 = 8 - 3 \times 2$$

$$1 = 3 - 2 \times 1$$

Extended Euclidean Algorithm

$$3 = 35 - 8 \times 4$$

$$2 = 8 - 3 \times 2$$

$$1 = 3 - 2 \times 1 = (35 - 8 \times 4) - (8 - 3 \times 2) \times 1$$

Extended Euclidean Algorithm

$$3 = 35 - 8 \times 4$$

$$2 = 8 - 3 \times 2$$

$$1 = 3 - 2 \times 1 = (35 - 8 \times 4) - (8 - 3 \times 2) \times 1$$
$$= (35 - 8 \times 4) - (8 - (35 - 8 \times 4) \times 2) \times 1$$

Extended Euclidean Algorithm

$$3 = 35 - 8 \times 4$$

$$2 = 8 - 3 \times 2$$

$$\begin{aligned} 1 &= 3 - 2 \times 1 = (35 - 8 \times 4) - (8 - 3 \times 2) \times 1 \\ &= (35 - 8 \times 4) - (8 - (35 - 8 \times 4) \times 2) \times 1 \\ &= 35 \times 3 - 8 \times 13 \end{aligned}$$

Extended Euclidean Algorithm

Given A,B > 0, set $x_1=1$, $x_2=0$, $y_1=0$, $y_2=1$,
$a_1=A$, $b_1=B$, $i=1$.

Repeat while $b_i>0$: {$i = i + 1$;
$q_i = a_{i-1}$ div b_{i-1}; $b_i = a_{i-1}-qb_{i-1}$; $a_i = b_{i-1}$;
$x_{i+1}=x_{i-1}-q_ix_i$; $y_{i+1}=y_{i-1}-q_iy_i$}.

For all i: $Ax_i + By_i = a_i$. Final $a_i = \gcd(A,B)$.

Digital Signatures

Recall that with RSA,

$D(E(Y)) = Y^{ED} \bmod N = Y$

$E(D(Y)) = Y^{DE} \bmod N = Y$

Only Alice (knowing the factorization of N) knows D. Hence only Alice can compute $D(Y) = Y^{D} \bmod N$.

This $D(Y)$ serves as Alice's signature on Y.

The Digital Signature Algorithm

In 1991, the National Institute of Standards and Technology published a Digital Signature Standard that was intended as an option free of intellectual property constraints.

The Digital Signature Algorithm

DSA uses the following parameters

- Prime p – anywhere from 512 to 1024 bits
- Prime q – 160 bits such that q divides p-1
- Integer h in the range $1 < h < p\text{-}1$
- Integer $g = h^{(p-1)/q} \bmod p$
- Secret integer x in the range $1 < x < q$
- Integer $y = g^x \bmod p$

The Digital Signature Algorithm

To sign a 160-bit message M,

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The pair (r,s) is the signature on M.

The Digital Signature Algorithm

A signature (r,s) on M is verified as follows:

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Accept the signature only if $v = r$.

Elliptic Curve Cryptosystems

Elliptic Curve Cryptosystems

An elliptic curve

Elliptic Curve Cryptosystems

An elliptic curve

$$y^2 = x^3 + Ax + B$$

Elliptic Curves

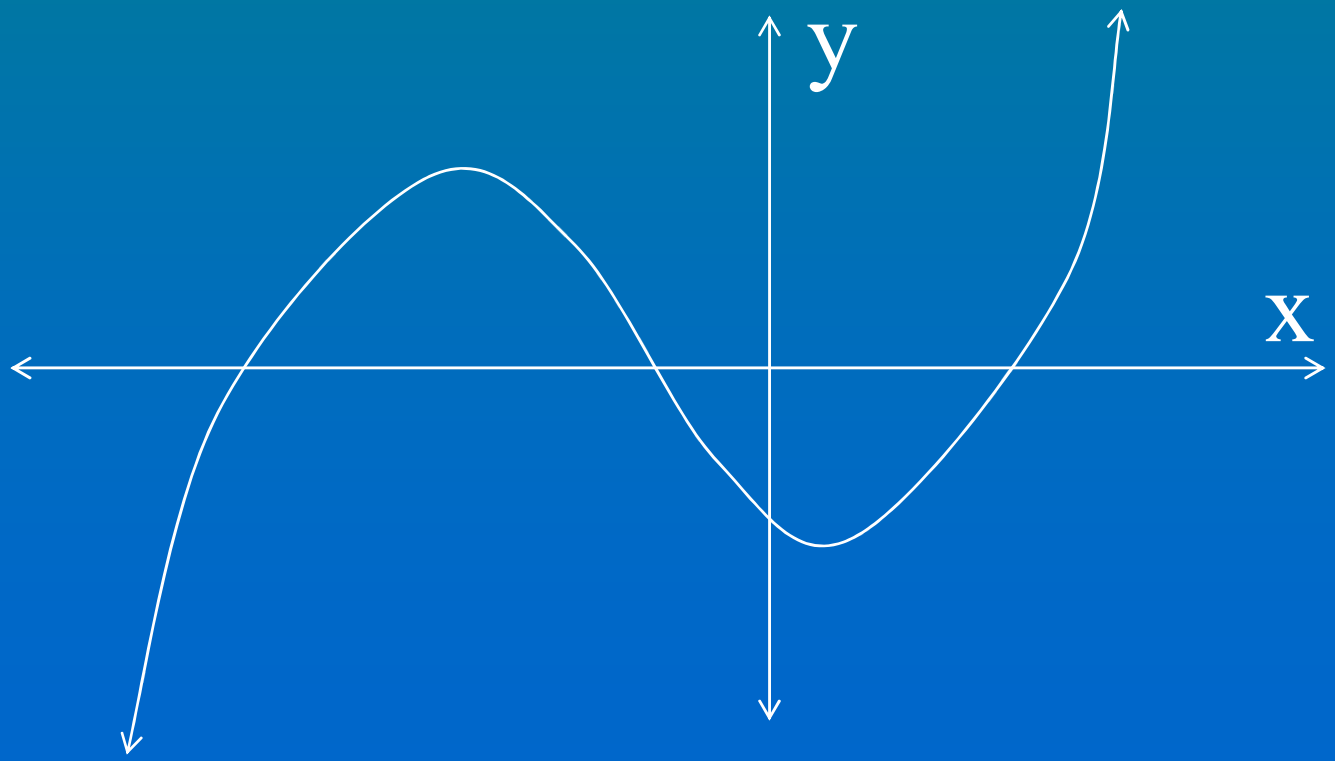
$$y^2 = x^3 + Ax + B$$

Elliptic Curves

$$y = x^3 + Ax + B$$

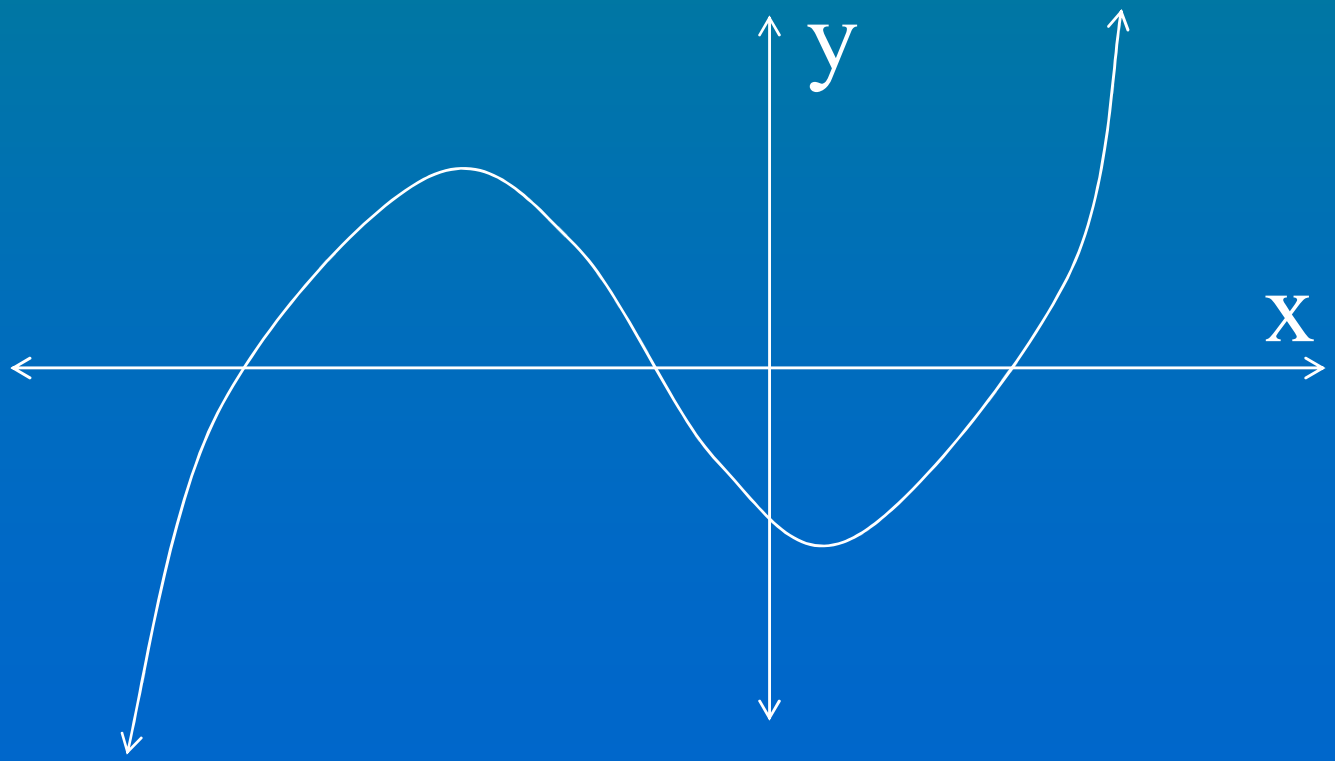
Elliptic Curves

$$y = x^3 + Ax + B$$



Elliptic Curves

$$y^2 = x^3 + Ax + B$$



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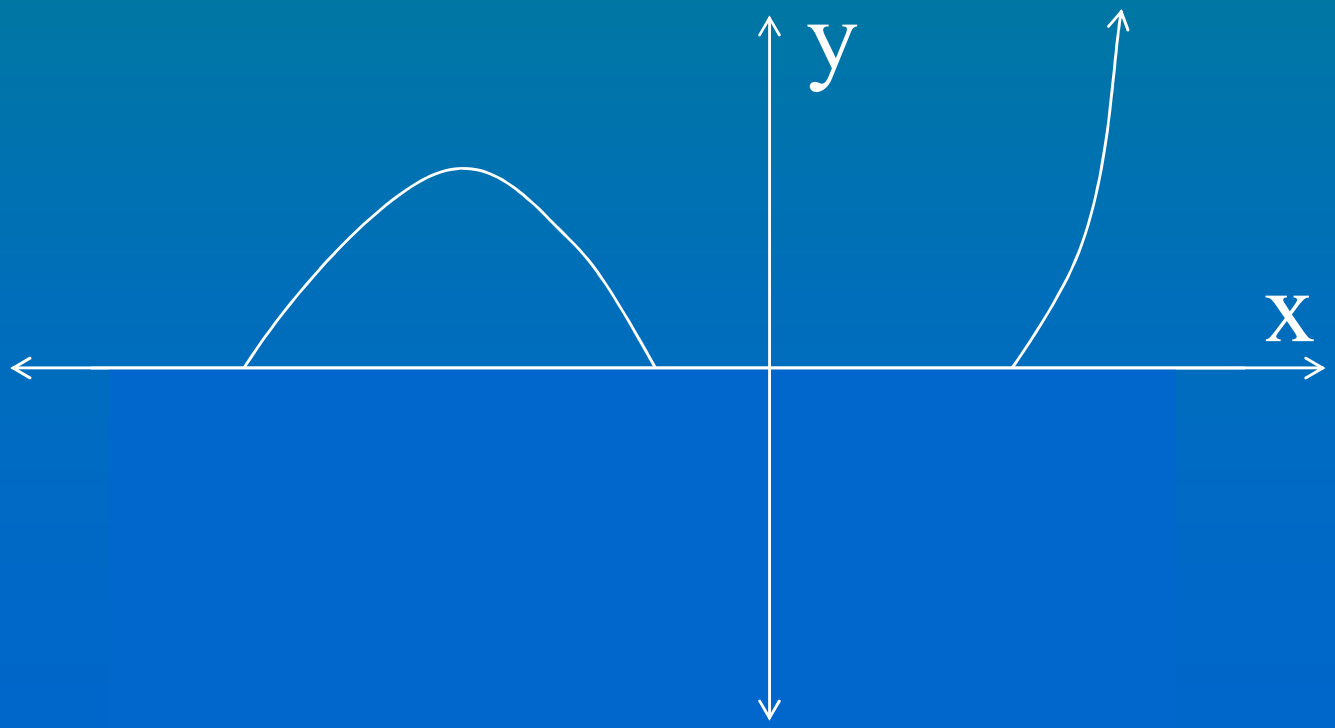

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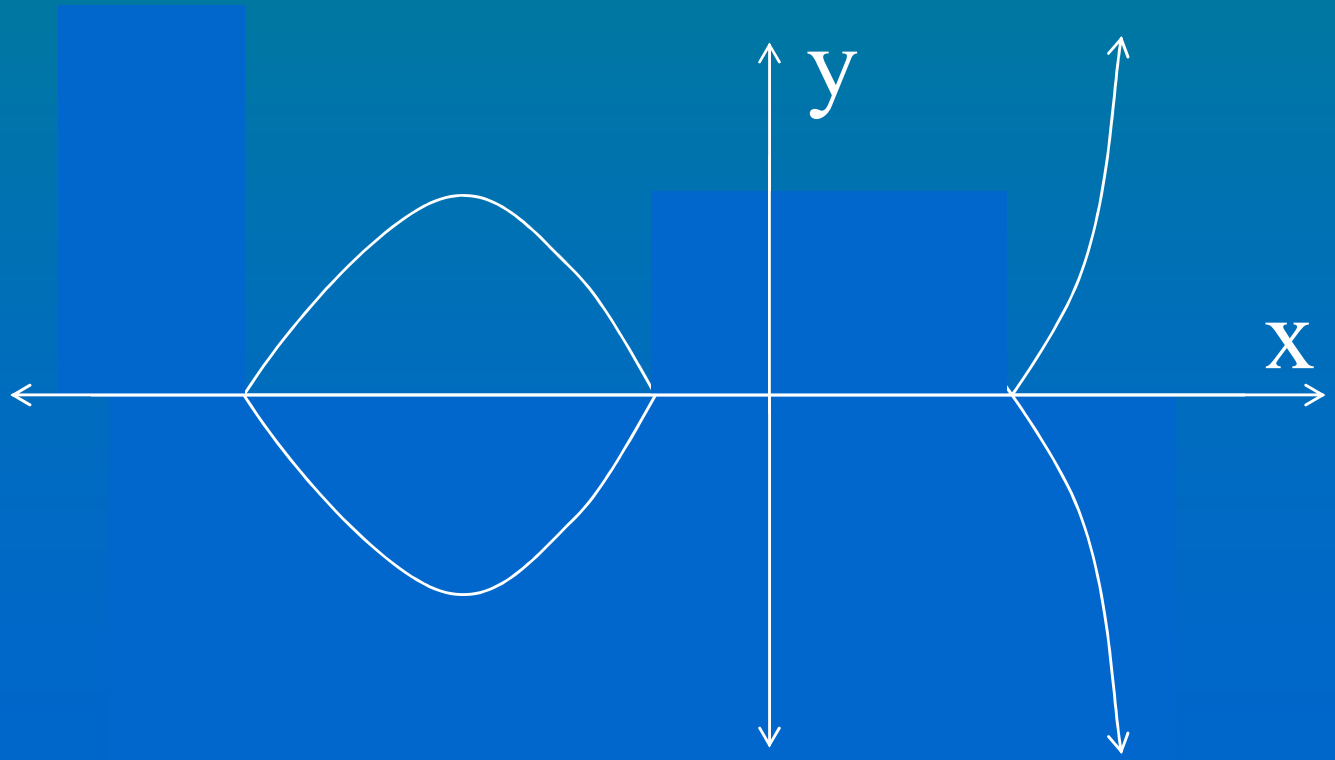

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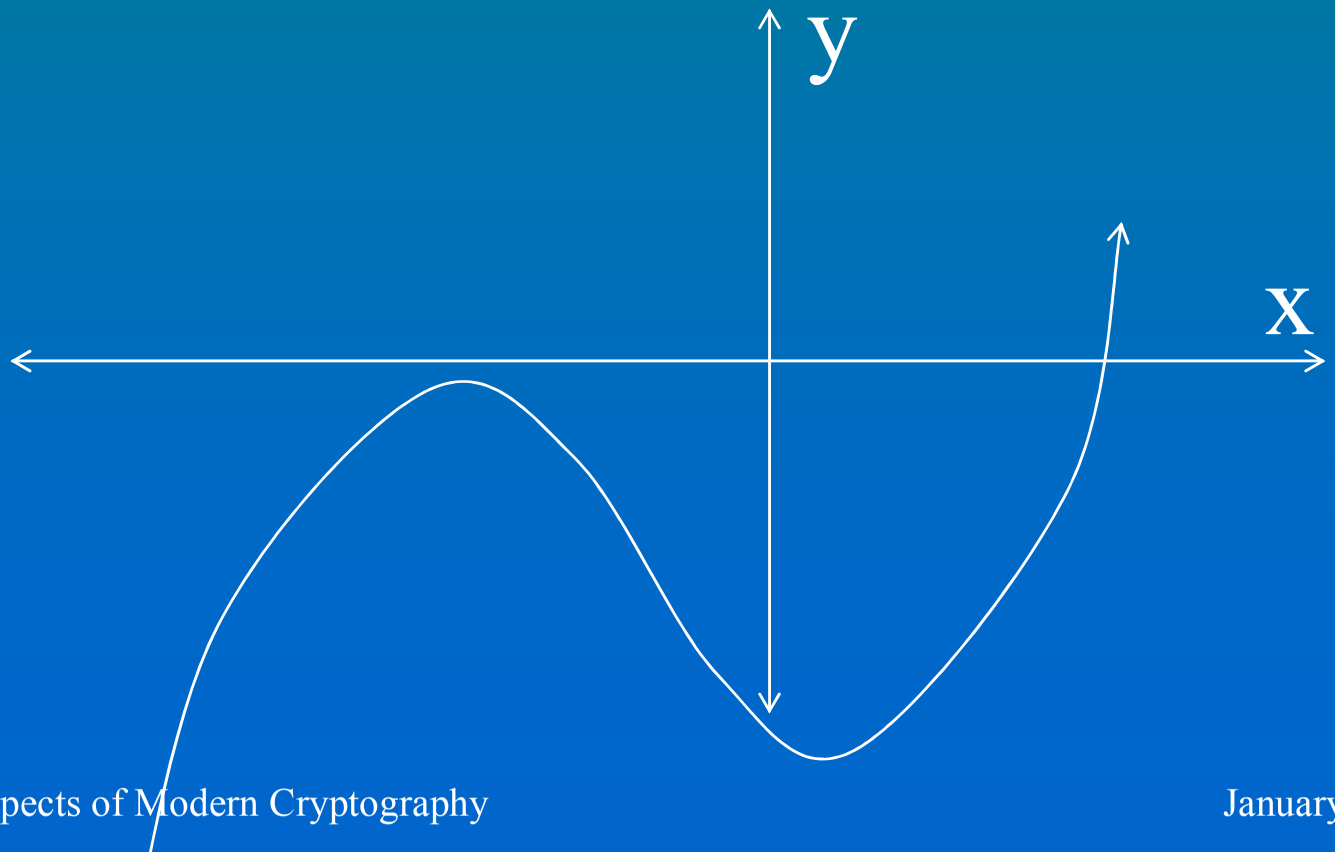

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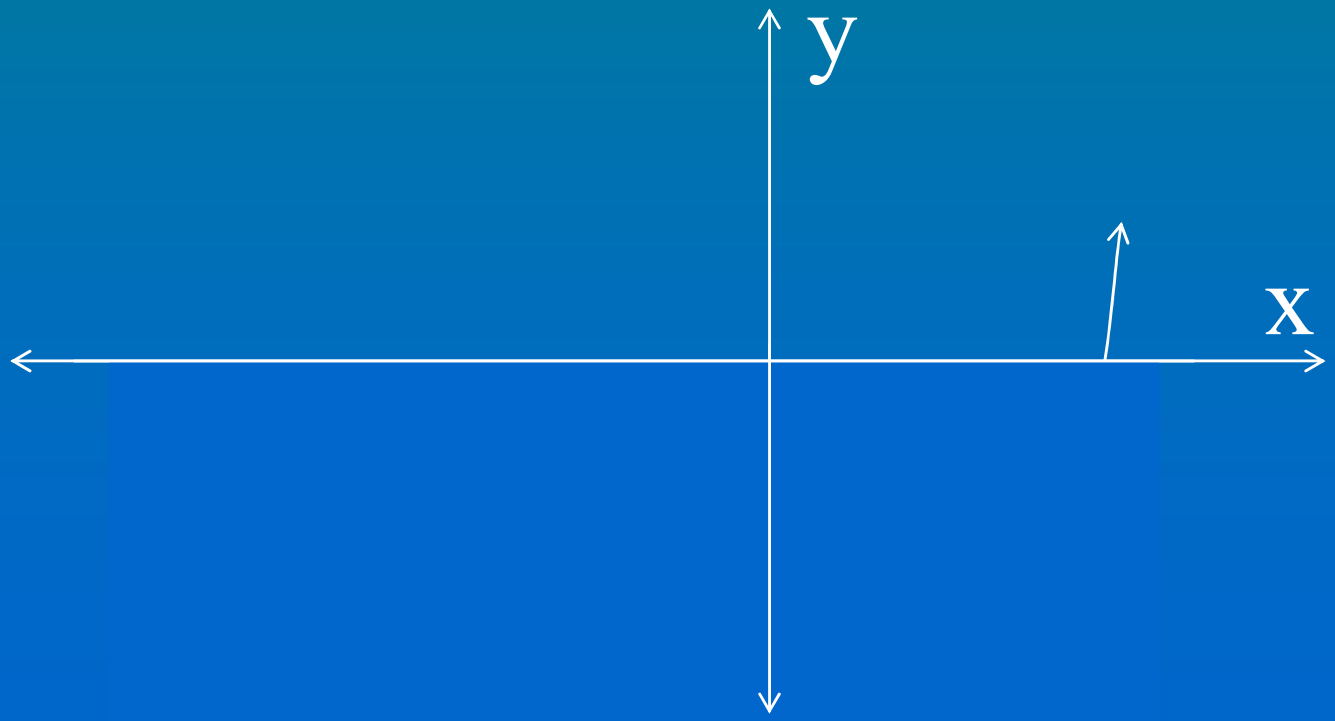

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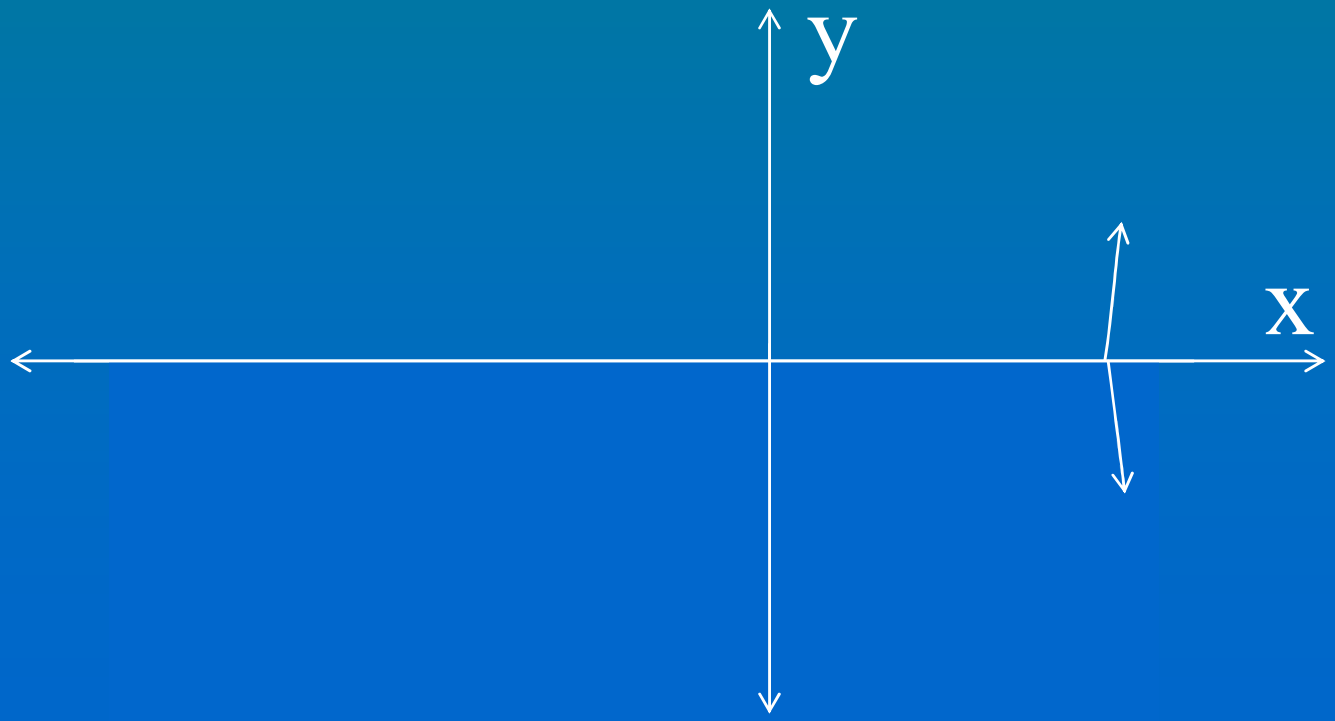

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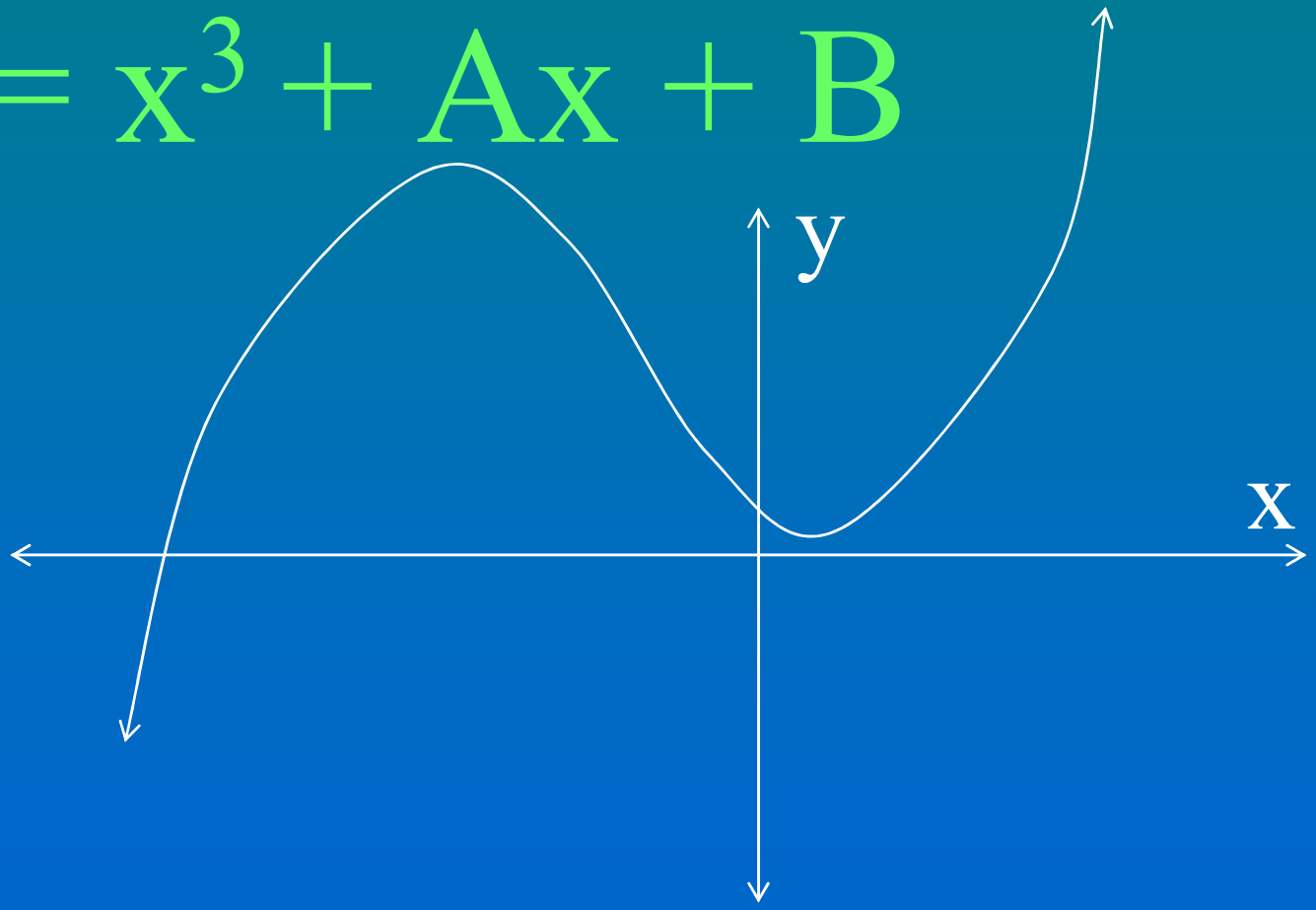

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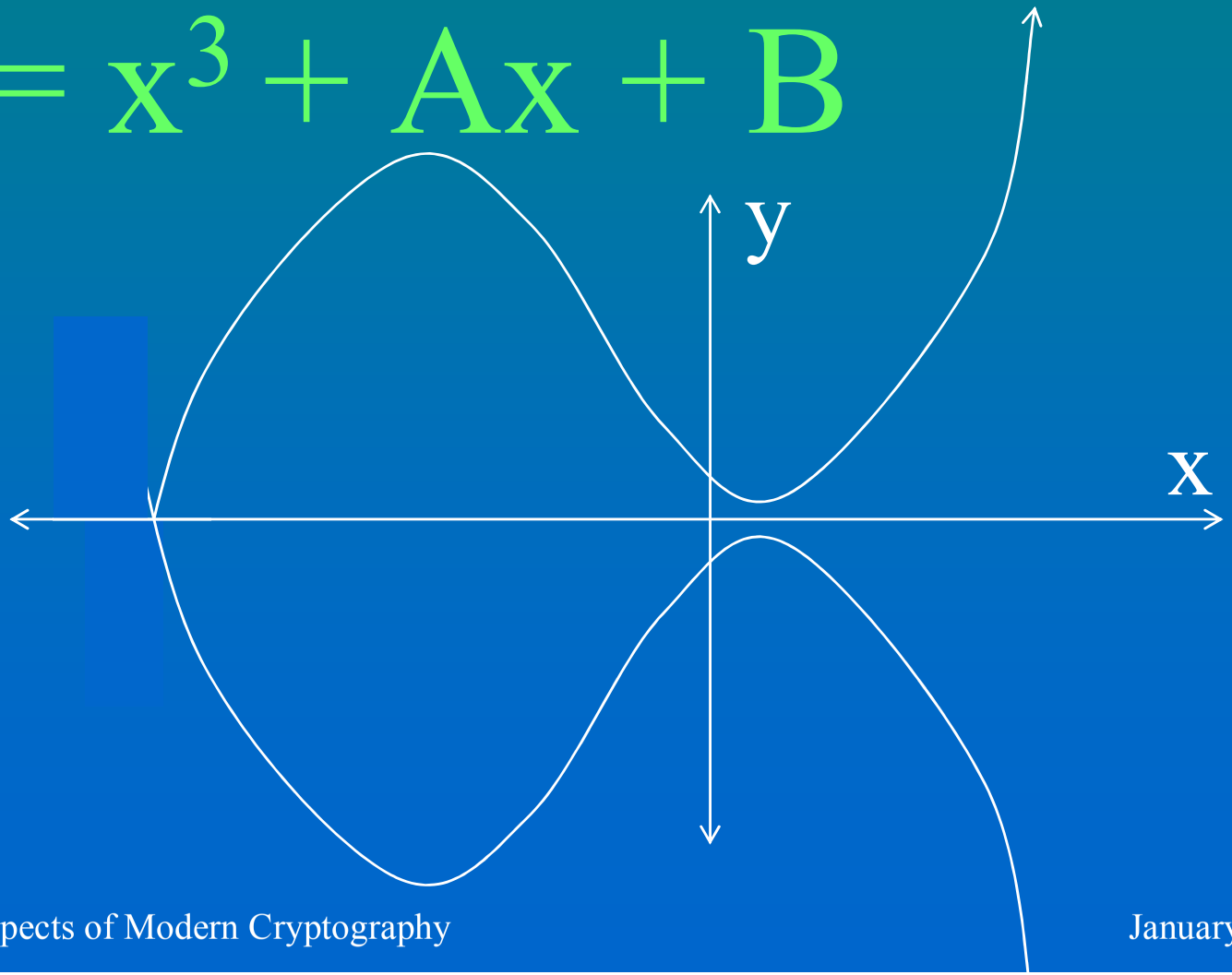
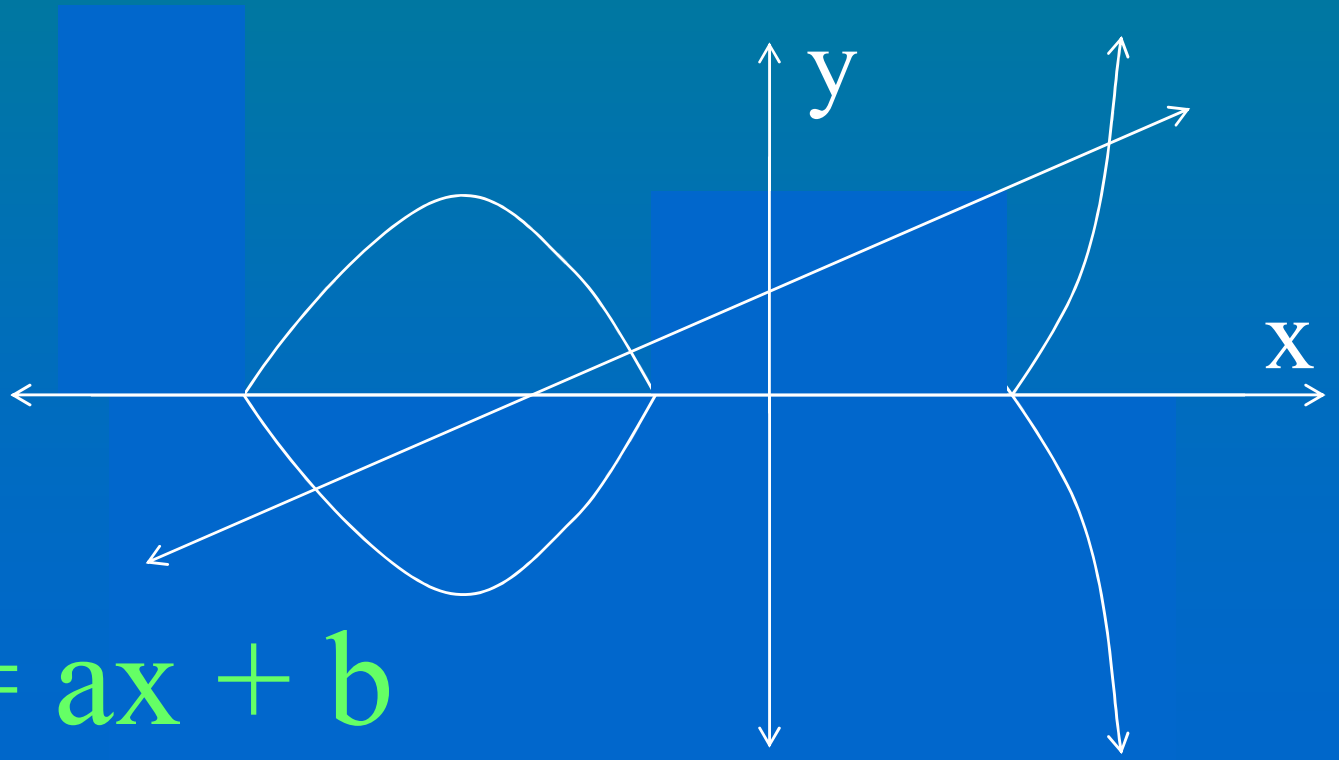


Elliptic Curves Intersecting Lines

$$y^2 = x^3 + Ax + B$$



$$y = ax + b$$

Elliptic Curves Intersecting Lines

Non-vertical Lines

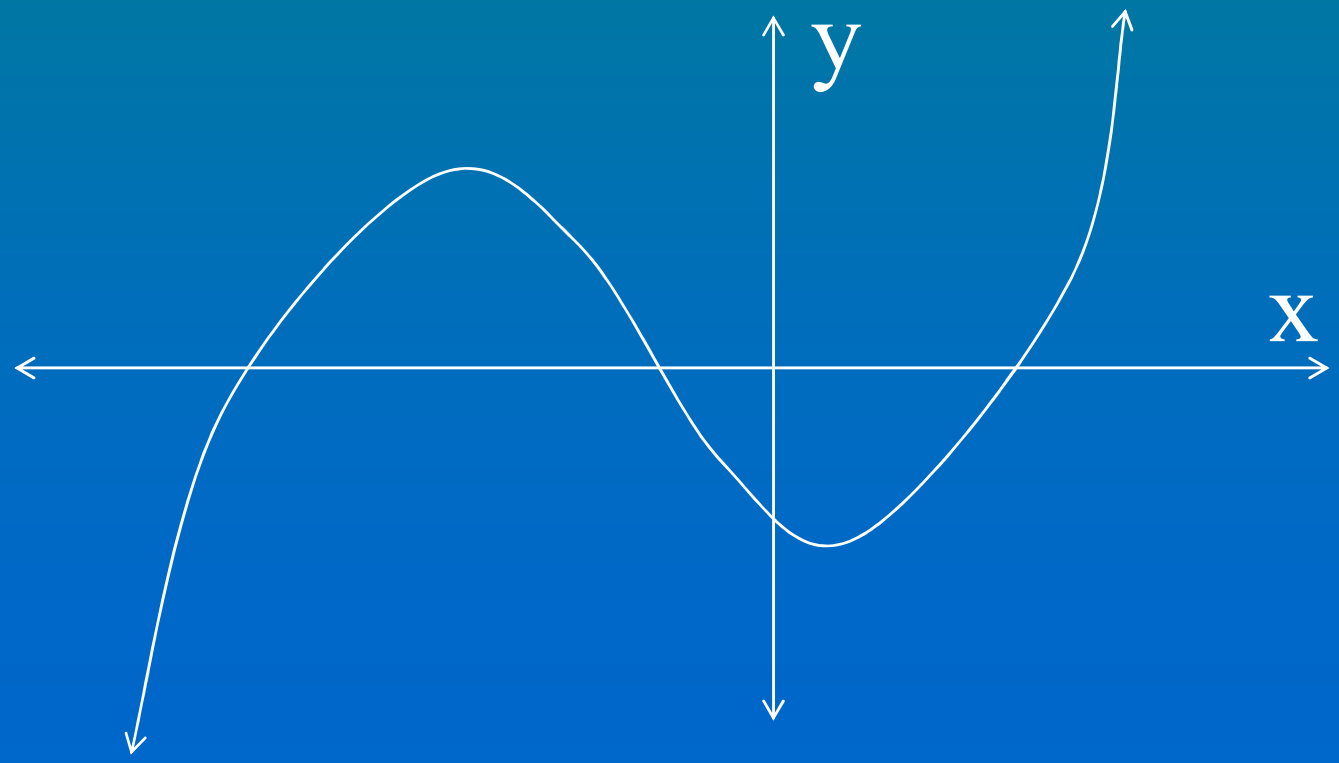
$$\begin{cases} y^2 = x^3 + Ax + B \\ y = ax + b \end{cases}$$

$$(ax + b)^2 = x^3 + Ax + B$$

$$x^3 + A'x^2 + B'x + C' = 0$$

Elliptic Curves Intersecting Lines

$$x^3 + A'x^2 + B'x + C' = 0$$



Elliptic Curves Intersecting Lines

Non-vertical Lines

- 1 intersection point (typical case)
- 2 intersection points (tangent case)
- 3 intersection points (typical case)

Elliptic Curves Intersecting Lines

Vertical Lines

$$\begin{cases} y^2 = x^3 + Ax + B \\ x = c \end{cases}$$

$$y^2 = c^3 + Ac + B$$

$$y^2 = C$$

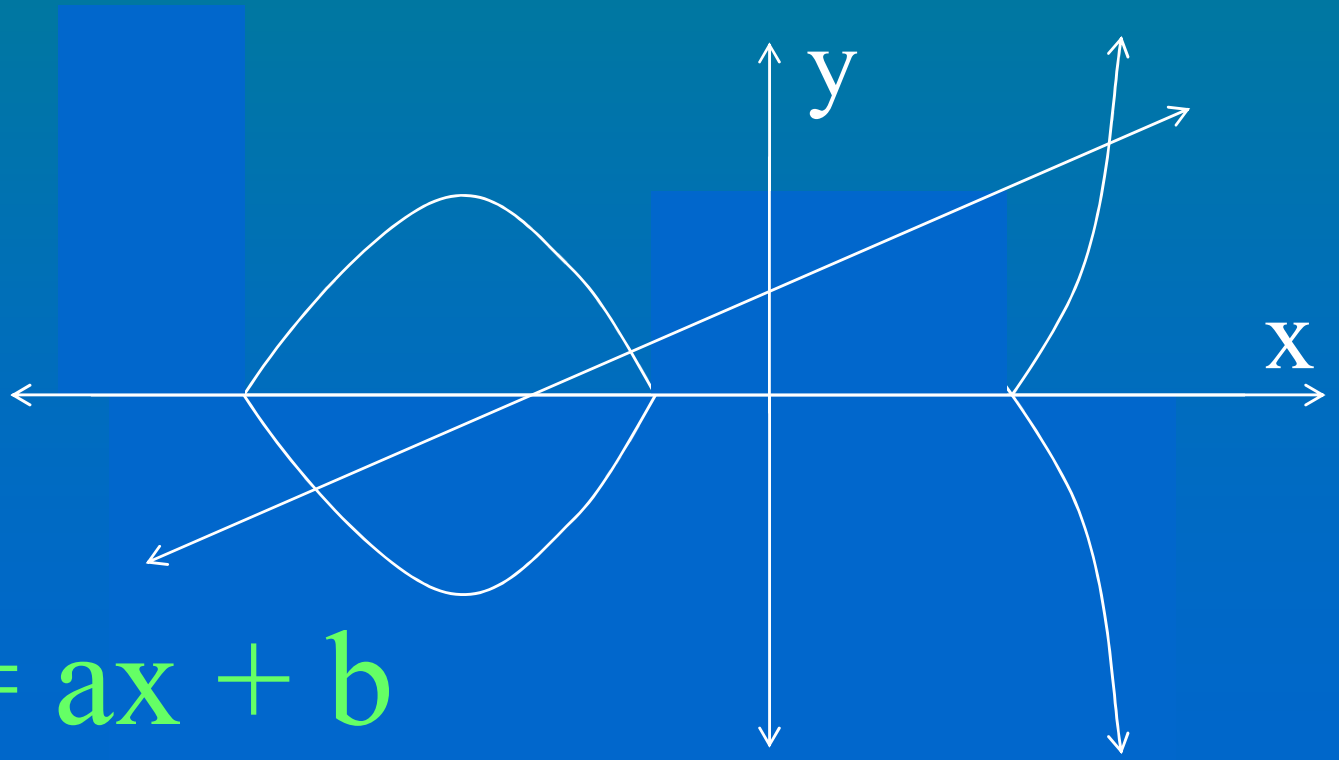
Elliptic Curves Intersecting Lines

Vertical Lines

- 0 intersection point (typical case)
- 1 intersection points (tangent case)
- 2 intersection points (typical case)

Elliptic Groups

$$y^2 = x^3 + Ax + B$$



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Elliptic Groups

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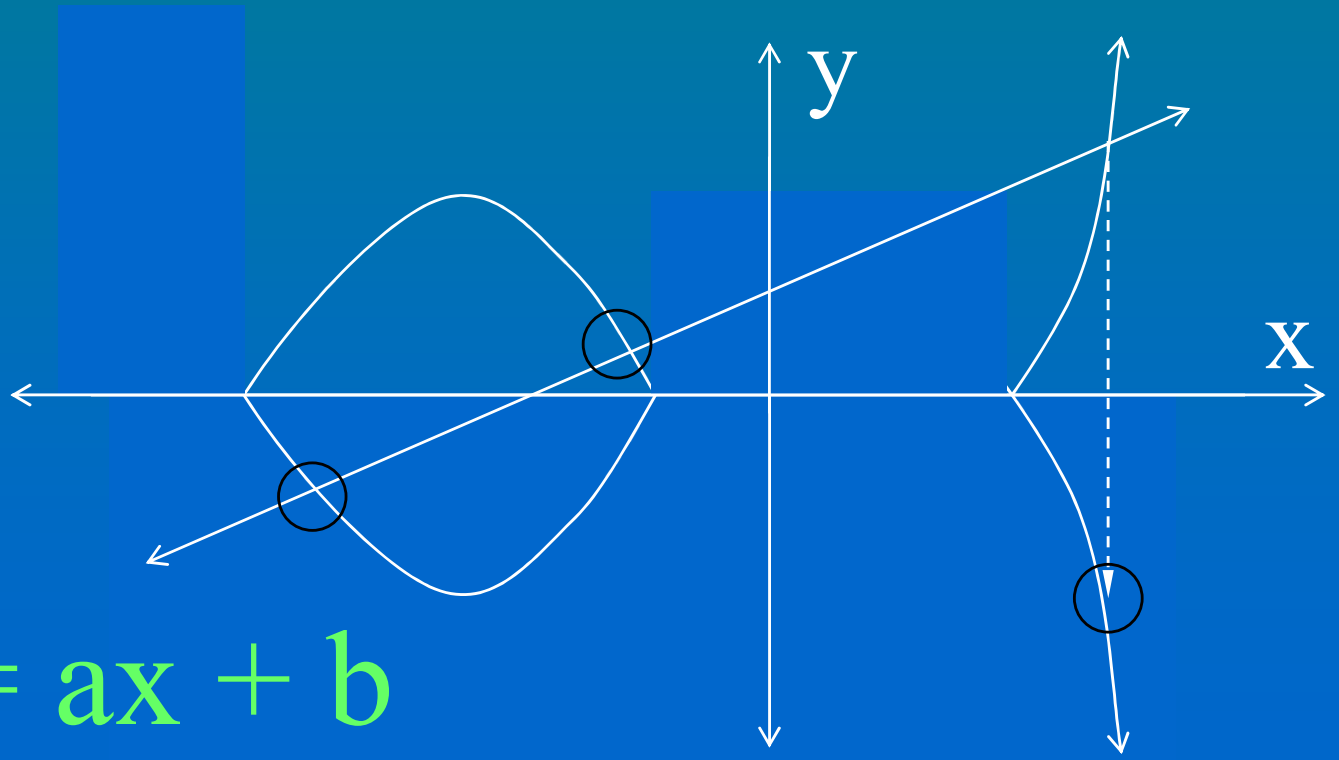

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Elliptic Groups

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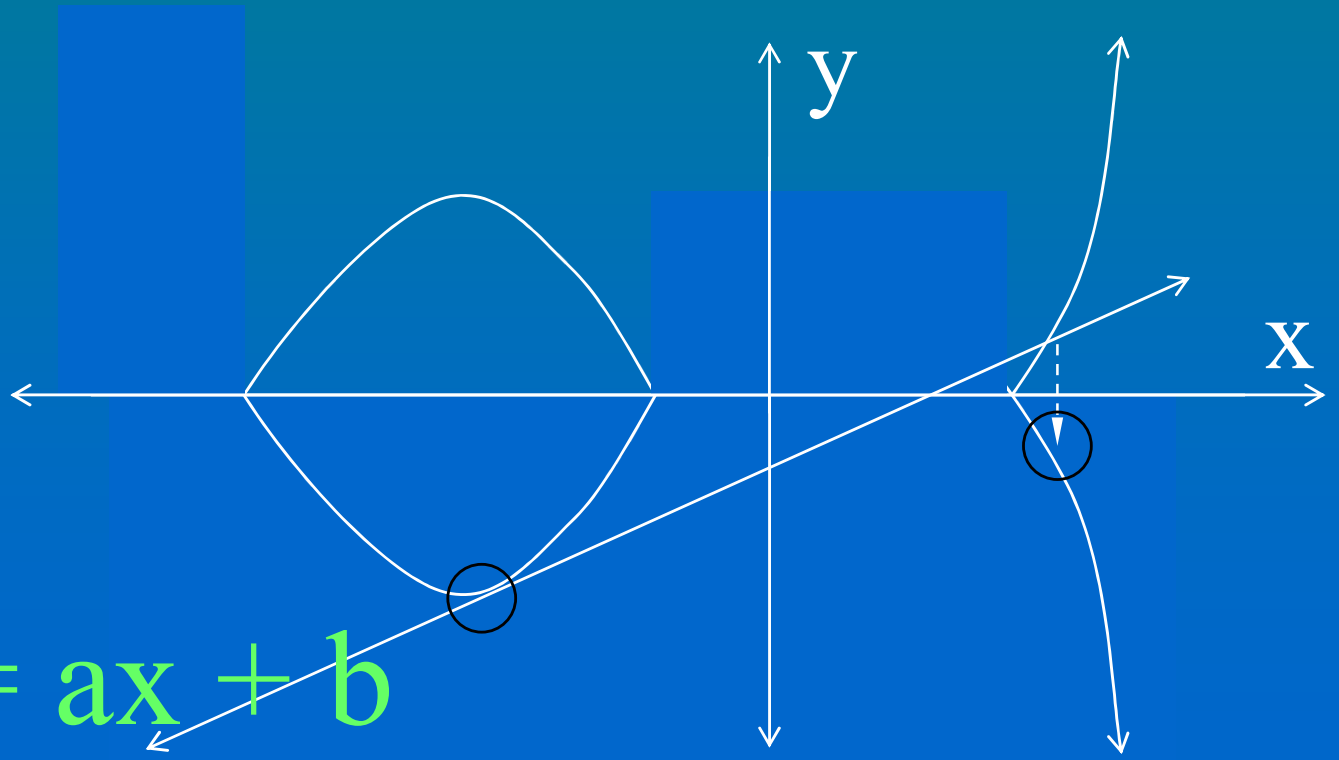

Elliptic Groups

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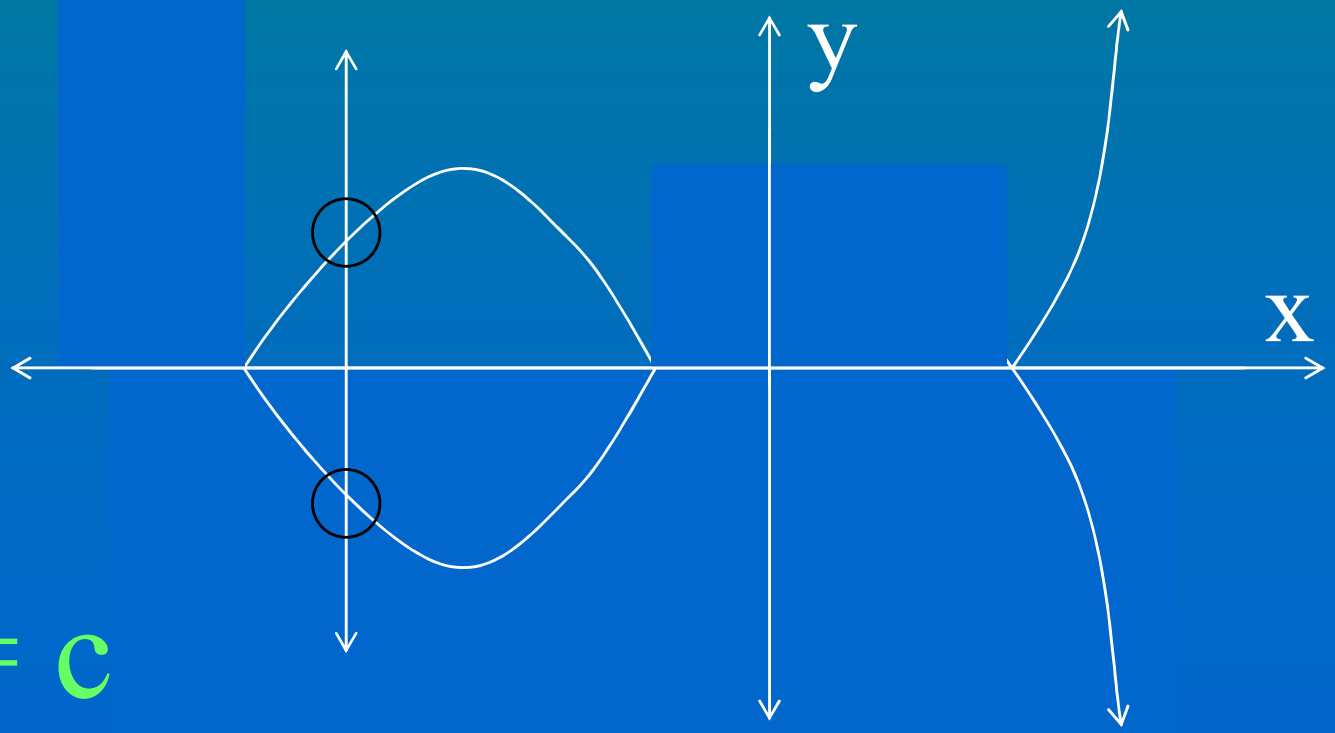


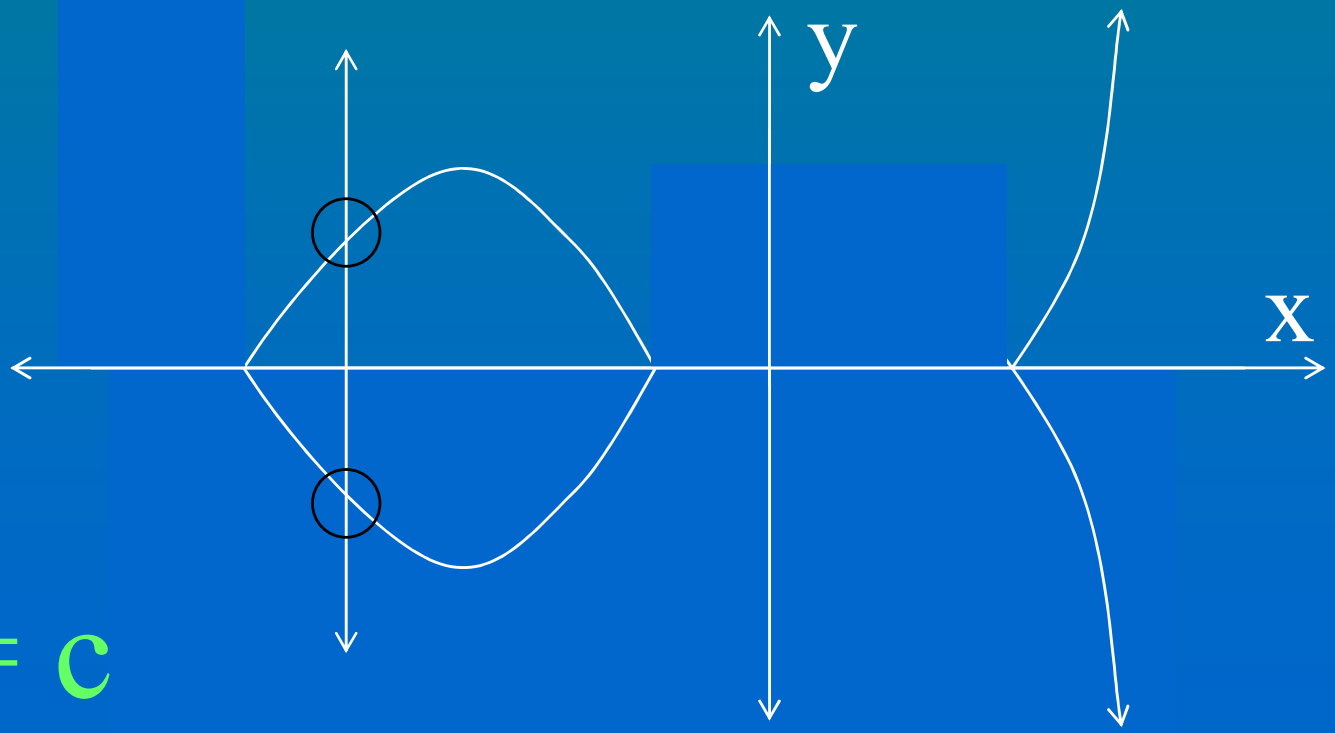
$$x = c$$

Elliptic Groups

- Add an “artificial” point I to handle the vertical line case.
- This point I also serves as the group identity value.

Elliptic Groups

$$y^2 = x^3 + Ax + B$$



$$x = c$$

Elliptic Groups

$$(x_1,y_1) \times (x_2,y_2) = (x_3,y_3)$$

$$x_3 = ((y_2-y_1)/(x_2-x_1))^2 - x_1 - x_2$$
$$y_3 = -y_1 + ((y_2-y_1)/(x_2-x_1))\ (x_1-x_3)$$

when $x_1 \neq x_2$

Elliptic Groups

$$(x_1,y_1) \times (x_2,y_2) = (x_3,y_3)$$

$$x_3 = ((3x_1^2+A)/(2y_1))^2 - 2x_1$$
$$y_3 = -y_1 + ((3x_1^2+A)/(2y_1)) (x_1-x_3)$$

when $x_1 = x_2$ and $y_1 = y_2 \neq 0$

Elliptic Groups

$$(x_1,y_1) \times (x_2,y_2) = I$$

when $x_1 = x_2$ but $y_1 \neq y_2$ or $y_1 = y_2 = 0$

$$(x_1,y_1) \times I = (x_1,y_1) = I \times (x_1,y_1)$$

$$I \times I = I$$

The Fundamental Equation

$$Z \equiv Y^X \bmod N$$

The Fundamental Equation

$$Z = Y^X \text{ in } E_p(A,B)$$

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When Z is unknown, it can be efficiently computed by repeated squaring.

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When X is unknown, this version of the discrete logarithm is believed to be quite hard to solve.

The Fundamental Equation

$$Z = Y^X \text{ in } E_p(A,B)$$

When Y is unknown, it *can* be efficiently computed by "sophisticated" means.

Diffie-Hellman Key Exchange

Alice	Bob
• Randomly select a large integer a and send A $= Y^a \bmod N$.	• Randomly select a large integer b and send B $= Y^b \bmod N$.
• Compute the key $K = B^a \bmod N$.	• Compute the key $K = A^b \bmod N$.

$$B^a = Y^{ba} = Y^{ab} = A^b$$

Diffie-Hellman Key Exchange

Alice

- Randomly select a large integer a and send A $= Y^a$ in E_p.
- Compute the key $K = B^a$ in E_p.

Bob

- Randomly select a large integer b and send B $= Y^b$ in E_p.
- Compute the key $K = A^b$ in E_p.

$$B^a = Y^{ba} = Y^{ab} = A^b$$

DSA on Elliptic Curves

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- Almost identical to DSA over the integers.

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- Almost identical to DSA over the integers.
- Replace operations mod p and q with operations in E_p and E_q.

Why use Elliptic Curves?

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- The best *currently known* algorithm for EC discrete logarithms would take about as long to find a 160-bit EC discrete log as the best *currently known* algorithm for integer discrete logarithms would take to find a 1024-bit discrete log.

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- The best *currently known* algorithm for EC discrete logarithms would take about as long to find a 160-bit EC discrete log as the best *currently known* algorithm for integer discrete logarithms would take to find a 1024-bit discrete log.
- 160-bit EC algorithms are somewhat faster and use shorter keys than 1024-bit "traditional" algorithms.

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- EC discrete logarithms have been studied far less than integer discrete logarithms.
- Results have shown that a fundamental break in integer discrete logs would also yield a fundamental break in EC discrete logs, although the reverse may not be true.
- Basic EC operations are more cumbersome than integer operations, so EC is only faster if the keys are *much* smaller.