Quantum Computing Problem Set 1

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Problem 1: A Qubit

The basic unit of quantum information is the qubit. In this problem we will discuss a single qubit system. You are given a qubit with a wave function given by \( |\psi\rangle = (\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}) |0\rangle + (\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}) |1\rangle \).

(a) What is the bra \( \langle \psi | \) corresponding to this state?

(b) Show that this state is a normalized wave function, that is \( \langle \psi | \psi \rangle = 1 \).

(c) If you measure this qubit in the computational basis \( |0\rangle, |1\rangle \), what are the probabilities of obtaining each of the two outcomes for this measurement?

(d) Define the two basis states \( |+\rangle = \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle \) and \( |-\rangle = \frac{1}{\sqrt{2}} |0\rangle - \frac{1}{\sqrt{2}} |1\rangle \). Calculate the inner product of these states with \( |\psi\rangle \). That is, find \( \langle +|\psi \rangle \) and \( \langle -|\psi \rangle \).

(e) Verify that \( |+\rangle \) and \( |-\rangle \) are orthogonal: \( \langle +|-\rangle = 0 \). Verify that \( |+\rangle \) and \( |-\rangle \) are also both properly normalized. In other words show that \( \langle +|+\rangle = \langle -|-\rangle = 1 \).

(f) Suppose we express \( |\psi\rangle \) in the \( |+\rangle \), \( |-\rangle \) basis as \( |\psi\rangle = \alpha |+\rangle + \beta |-\rangle \). Show that \( \alpha = \langle +|\psi \rangle \) and \( \beta = \langle -|\psi \rangle \). Hint: form the bracket \( \langle +|\psi \rangle \) and use the fact that \( |+\rangle \) and \( |-\rangle \) are orthogonal.

(g) Use the previous three parts to express \( |\psi\rangle \) as \( |\psi\rangle = \alpha |+\rangle + \beta |-\rangle \) but with complex numbers instead of \( \alpha \) and \( \beta \).

(h) If we measure the state \( |\psi\rangle \) in the \( |+\rangle \) and \( |-\rangle \) basis, what are the probabilities of obtaining each of these two outcomes?

Problem 2: Single Qubit Matrices

The basic manipulations of a qubit will be two by two matrices. In this problem we will become familiar a particular single qubit matrix.

(a) Let

\[
U = \begin{bmatrix}
\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\
\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{bmatrix}
\]

Calculate the complex conjugate to \( U \): \( U^\dagger \). Your answer will be a two by two matrix.

(b) Calculate the adjoint to \( U \): \( U^\dagger \). Again your answer will be a two by two matrix.

(c) Verify that \( U \) is unitary. That is, show that \( UU^\dagger = I \). This means that \( U \) is a valid quantum evolution.

(d) If we apply the evolution corresponding to \( U \) to \( |\psi\rangle \), our new state is \( |\psi'\rangle = U|\psi\rangle \). What is \( |\psi'\rangle \) if \( |\psi\rangle = (\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}) |0\rangle + (\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}) |1\rangle \)?

(e) Calculate \( U^2 \) and \( U^4 \). One of these should be negative the identity matrix.

(f) Suppose we apply the evolution corresponding to \( U \) twice to a qubit with the wave function \( |\psi\rangle = (\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}) |0\rangle + (\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}) |1\rangle \). What is the state after this application?

(g) Recall that the Pauli matrices are the following two by two matrices:

\[
\sigma_0 = I = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}, \quad \sigma_1 = X = \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}, \quad \sigma_2 = Y = \begin{bmatrix}
0 & -i \\
i & 0
\end{bmatrix}, \quad \text{and,} \quad \sigma_3 = Z = \begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix}
\]

Express \( U \) as a linear combination of these matrices. That is, express \( U \) as

\[
U = s_0 \sigma_0 + s_1 \sigma_1 + s_2 \sigma_2 + s_3 \sigma_3
\]

for specific, complex \( s_i \)'s.

(h) Calculate \( UU^\dagger \). Your answer will be a two by two matrix. In particular it will be equal to one of the Pauli matrices.

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