Quantum Computing
Problem Set 5 Solutions

1. (a) Many ways to check this but it should be obvious that the answer is $s_1 = 1, s_2 = 1$.
(b) 
$$|v⟩ = \frac{1}{2}(|000⟩ + |010⟩ + |100⟩ + |110⟩)$$
and discarding all the 0 terms
$$|v'⟩ = U|v⟩ = \frac{1}{2}(U|000⟩ + U|010⟩ + U|100⟩ + U|110⟩)$$
$$= \frac{1}{2}(|0, 0, 0 ⊕ f(0, 0)| + |0, 1, 0 ⊕ f(0, 1)| + |1, 0, 0 ⊕ f(1, 0)| + |1, 1, 0 ⊕ f(1, 1)|)$$
$$= \frac{1}{2}(|000⟩ + |011⟩ + |101⟩ + |110⟩)$$
(c) 
$$P_0 = I ⊗ I ⊗ |0⟩⟨0|$$
$$= |000⟩⟨000| + |010⟩⟨010| + |100⟩⟨100| + |110⟩⟨110|$$
$$P_0|v'⟩ = \frac{1}{2}(|000⟩ + |110⟩)$$
$$Pr[0] = ⟨v'|\frac{1}{2}(|000⟩ + |110⟩)$$
$$= \frac{1}{2}⟨000⟩ + 011⟩ + 101⟩ + 110⟩|000⟩ + |110⟩$$
$$= \frac{1}{4}(1 + 1) = \frac{1}{2}$$
When we get outcome 0, the new wave function will be
$$\frac{\frac{1}{2}(|000⟩ + |110⟩)}{\sqrt{\frac{1}{2}}} = \frac{1}{\sqrt{2}}(|000⟩ + |110⟩) = \frac{1}{\sqrt{2}}(|00⟩ + |11⟩) ⊗ |0⟩$$
Therefore the state of the first two qubits is $\frac{1}{\sqrt{2}}(|00⟩ + |11⟩)$.
(d) 
$$H ⊗ H = \frac{\frac{1}{2}}{1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$
\[
H \otimes H(\frac{1}{\sqrt{2}}(\ket{00} + \ket{11})) = \frac{1}{\sqrt{2}}(\ket{00} + \ket{11})
\]

If we now measure in the computational basis, we get

\[
Pr[00] = ||\langle 00 | \frac{1}{\sqrt{2}}(\ket{00} + \ket{11}) ||^2 = \frac{1}{2}
Pr[01] = 0
Pr[10] = 0
Pr[11] = 1
\]

The two possible outcomes are 00 and 11, with probability 1/2.

(e) Since \(s_1 + s_2 = 0 \mod 2\), the only non-zero solution is \(s_1 = 1, s_2 = 1\), the same as in part a.

2. (a)

\[
\begin{align*}
 f_{0,0}(x_1, x_2) &= (1 - x_1)(1 - x_2) \\
 f_{0,1}(x_1, x_2) &= (1 - x_1)x_2 \\
 f_{1,0}(x_1, x_2) &= x_1(1 - x_2) \\
 f_{1,1}(x_1, x_2) &= x_1x_2
\end{align*}
\]

(b) By discarding the 0 terms we get

\[
U_{1,0}|011\rangle = |0, 1, 1 \oplus f_{1,0}(0, 1)\rangle = |0, 1, 1 \oplus 0\rangle = |011\rangle
\]

(c)

\[
|v'\rangle = U_{\alpha_1, \alpha_2}|v\rangle = U_{\alpha_1, \alpha_2} \frac{1}{2\sqrt{2}} \sum_{x_1=0}^{1} \sum_{x_2=0}^{1} (|x_1, x_2, 0\rangle - |x_1, x_2, 1\rangle)
\]

\[
= \frac{1}{2} \sum_{x_1=0}^{1} \sum_{x_2=0}^{1} U_{\alpha_1, \alpha_2}(|x_1, x_2\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle))
\]

If \(x_1, x_2 = \alpha_1, \alpha_2\), then \(f_{\alpha_1, \alpha_2}(x_1, x_2) = 1\). Effectively this will flip the third qubit, so the inner term will be

\[
|x_1, x_2\rangle \otimes \frac{1}{\sqrt{2}}(|1\rangle - |0\rangle)
\]

In the rest of the cases, the inner term remains

\[
|x_1, x_2\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)
\]
Suppose we have two possible pairs of \((a_1, a_2)\). For example, you can consider the unitary

\[ (-1)^{f_{a_1, a_2}(x_1, x_2)} |x_1, x_2\rangle \otimes \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle) \]

and the result follows.

You can also get this more directly by applying the phase kickback argument.

(d) Suppose we have two possible pairs of \((a_1, a_2)\); call these \((a_1, a_2)\) and \((a'_1, a'_2)\). Then we form the inner product:

\[
\langle a'_1, a'_2 | a_1, a_2 \rangle = \left[ \frac{1}{2} \sum_{x_1, x_2 = 0} ( -1)^{f_{a_1, a_2}(x'_1, x'_2)} \langle x'_1, x'_2 | x_1, x_2 \rangle \right] \left[ \frac{1}{2} \sum_{x_1, x_2 = 0} ( -1)^{f_{a_1, a_2}(x_1, x_2)} |x_1, x_2\rangle \right]
\]

\[
= \frac{1}{4} \sum_{x'_1, x'_2 = 0} \sum_{x_1, x_2 = 0} \langle x'_1, x'_2 | x_1, x_2 \rangle \sum_{x_1, x_2 = 0} ( -1)^{f_{a_1, a_2}(x'_1, x'_2) + f_{a_1, a_2}(x_1, x_2)} \langle x'_1, x'_2 | x_1, x_2 \rangle
\]

\[
= \frac{1}{4} \sum_{x'_1, x'_2 = 0} \sum_{x_1, x_2 = 0} ( -1)^{2f_{a_1, a_2}(x_1, x_2)} \delta(x'_1, x'_2, x_1, x_2)
\]

Thus we have 0 when \(x' \neq x\) and 1 when \(x' = x\). So we have an orthonormal basis.

(e) There are many ways to derive this unitary. For example, you can consider the unitary elements as unknowns and solve the equations or work backwards and eyeball the unitary that satisfies the requirements.

You should end up with

\[
U = \frac{1}{2} \begin{bmatrix}
-1 & 1 & 1 & 1 \\
1 & -1 & 1 & 1 \\
1 & 1 & -1 & 1 \\
1 & 1 & 1 & -1
\end{bmatrix}
\]

Let’s verify the four different cases

i. If \((\alpha_1, \alpha_2) = (0, 0)\), then \(|\alpha_1, \alpha_2\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)\).

\[ U|\alpha_1, \alpha_2\rangle = |00\rangle \]

ii. If \((\alpha_1, \alpha_2) = (0, 1)\), then \(|\alpha_1, \alpha_2\rangle = \frac{1}{2}(|00\rangle - |01\rangle + |10\rangle + |11\rangle)\).

\[ U|\alpha_1, \alpha_2\rangle = |01\rangle \]

iii. If \((\alpha_1, \alpha_2) = (1, 0)\), then \(|\alpha_1, \alpha_2\rangle = \frac{1}{2}(|00\rangle + |01\rangle - |10\rangle + |11\rangle)\).

\[ U|\alpha_1, \alpha_2\rangle = |10\rangle \]

iv. If \((\alpha_1, \alpha_2) = (1, 1)\), then \(|\alpha_1, \alpha_2\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle - |11\rangle)\).

\[ U|\alpha_1, \alpha_2\rangle = |11\rangle \]