1. Transition system $M$ with fairness constraints

**Solutions:**

a) Maximal SCCs

Omitting trivial SCCs, we have two maximal SCCs:

i) $S_1, S_4, S_8$

ii) $S_2, S_5, S_6, S_7, S_9, S_{10}, S_{11}, S_{12}$

If we include trivial SCCs, then there are four (add $S_0$ and $S_3$).

b) Does $M, S_0 \models E_c G p$ hold?

Holds:

First, $p$ is true in all states by definition.
The SCC $S_2, S_5, S_6, S_7, S_9, S_{10}, S_{11}, S_{12}$ satisfies all the fairness constraints.
We can reach the latter SCC from state $S_0$.
Thus, $M, S_0 \models E_c G p$ holds.

c) $p$ false in $S_6$. Does $M, S_0 \models E_c G p$ hold?

Does not hold:

Now there are three SCCs in the system: $S_2, S_5; S_7, S_{11}, S_{12}; S_1, S_4, S_8$
None of these SCCs satisfy the fairness constraints, so there are no fair SCCs.
Thus, $M, S_0 \models E_c G p$ does not hold.

2. The Spring Kripke structure

**Solutions:**

LTL formulae and run $\pi = S_1 S_2 S_1 S_2 S_3 S_3 S_3 \ldots$

a) extended

Not satisfied because ‘extended’ is false in $S_1$.

b) X extended

Satisfied because ‘extended’ is true at $S_2$.

c) XX extended

Not satisfied because ‘extended’ is false in $S_1$.

d) F extended

Satisfied because ‘extended’ is true in some future state (e.g. $S_2$).

e) G extended

Not satisfied because ‘extended’ is not true at all states in the path (e.g. $S_1$).
f) FG extended  
Satisfied because after we reach S_3, ‘extended’ is true globally.

g) !extended U malfunction  
Not satisfied because ‘extended’ is not continuously false until ‘malfunction’ is true –  
since ‘extended’ is true in S_2.

**LTL formulae and system Spring**  
*Note: by definition, a path π for LTL must be infinite.*

a) F extended  
Satisfied because from S_1 we must move to a state were extended is true in all paths.

b) G(!extended ⇒ X extended)  
Satisfied because !extended is true only at S_1, and the next state after S_1 in all paths must  
be S_2, at which extended holds.

c) FG extended  
Not satisfied because in path S_1 → S_2 → S_1 → S_2 … is it never the case that G extended  
is true.

d) !FG extended  
Not satisfied because in path S_1 → S_2 → S_3 → S_3 → S_3 … G extended is satisfied.

e) G(extended ⇒ X !extended)  
Not satisfied, look at path S_1 → S_2 → S_3 → S_3 → S_3 …

3. LTL equivalences  
**Solutions:**

a) XFp, FXp  
Equivalent

b) (FGp) ∧ (FGq), F(Gp ∧ Gq)  
Equivalent

c) (p U q) U q, p U q  
Equivalent

d) (p U q) ∧ (q U r), (p U r)  
Not equivalent. This is so because the first formula requires that q hold at some point in  
the path, while the second formula does not. Thus, a counterexample structure is:

Let p be true in state S_1, r true in state S_2, and q true in state S_3  
Then S_1 → S_1 → S_1 → S_2 satisfies the second formula but not the first.
e) Find a CTL* path formula

$$A[G(p \rightarrow X \neg p) \ G(\neg p \rightarrow Xp)]$$

4. Monotone functions and fixed points

**Solutions:**

a) Which are monotone?

H1 is monotone

Its is clear to see that if $$X_1 \subseteq X_2$$, then $$H_1(X_1) \subseteq H_1(X_2)$$ because $$H_1$$ will have removed the same elements from both $$X_1$$ and $$X_2$$, thereby maintaining their relationship.

H2 is not monotone

Counterexample:
Let $$X_1 = \{2\}$$, $$X_2 = \{2,5\}$$, then $$X_1 \subseteq X_2$$, but $$H_2(X_1) = \{5,9\}$$, $$H_2(X_2) = \{9\}$$, so $$H_2(X_1) \not\subseteq H_2(X_2)$$

H3 is monotone

A union with a larger set can either make the intersection larger, or make no change in size – it will never reduce the size. Thus the relation between $$X_1$$ and $$X_2$$ is maintained.

b) Greatest and least fixed points of $$H_3$$

Least fixed point: $$\{2,4\}$$

Greatest fixed point: $$\{1,2,3,4,5\}$$

c) Fixed points of $$H_2$$?

No fixed points because if the input contains any of $$\{2,5,9\}$$, those elements will be removed, resulting in a different output. Likewise, if the input does not contain any of $$\{2,5,9\}$$, then the output will be $$\{2,5,9\}$$ which will be different than the input.

5. Relational mu-calculus

**Solution:**

We need to prove $$p \models vZ.Z$$ so here $$f = Z$$

We use induction on $$m$$

Base case $$m = 0$$

$$v_0Z.f = 1$$ (by definition)

By mu-calculus grammar definition, $$p \models 1$$ so the base case holds.

Suppose this holds for $$m = p$$.

$$p \models V_ZZ$$
Then we compute $V_{p+1}Z.Z$

$V_{p+1}Z.Z = V_pZ.Z$ (replace $Z$ with $V_{p+1}Z.Z$ according to $vZ.f$ definition.)

But we know from the induction hypothesis that $p \models V_pZ.Z$. So $p \models V_{p+1}Z.Z$

And by induction for all $m \geq 0$, $p \models vZ.Z$