CSEP 573: Artificial Intelligence

Markov Decision Processes (MDPs)

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Many slides over the course adapted from Ali Farhadi, Dan Weld, Dan Klein, Stuart Russell or Andrew Moore
Outline (roughly next two weeks)

- Markov Decision Processes (MDPs)
  - MDP formalism
  - Value Iteration
  - Policy Iteration

- Reinforcement Learning (RL)
  - Relationship to MDPs
  - Several learning algorithms
Review: Expectimax

- What if we don’t know what the result of an action will be? E.g.,
  - In solitaire, next card is unknown
  - In minesweeper, mine locations
  - In pacman, the ghosts act randomly

- Can do **expectimax search**
  - Chance nodes, like min nodes, except the outcome is uncertain
  - Calculate **expected utilities**
  - Max nodes as in minimax search
  - Chance nodes take average (expectation) of value of children

- Today, we’ll learn how to formalize the underlying problem as a **Markov Decision Process**
Reinforcement Learning

- **Basic idea:**
  - Receive feedback in the form of *rewards*
  - Agent’s utility is defined by the reward function
  - Must learn to act so as to **maximize expected rewards**
Reinforcement Learning

https://www.youtube.com/watch?v=W_gxLKSsSIE
Grid World

- The agent lives in a grid
- Walls block the agent’s path
- The agent’s actions do not always go as planned:
  - 80% of the time, the action North takes the agent North (if there is no wall there)
  - 10% of the time, North takes the agent West; 10% East
  - If there is a wall in the direction the agent would have been taken, the agent stays put
- Small “living” reward each step
- Big rewards come at the end
- Goal: maximize sum of rewards
Grid World Actions

Deterministic

Stochastic
Markov Decision Processes

- An MDP is defined by:
  - A set of states $s \in S$
  - A set of actions $a \in A$
  - A transition function $T(s,a,s')$
    - Prob that $a$ from $s$ leads to $s'$
    - i.e., $P(s' | s,a)$
    - Also called the model
  - A reward function $R(s, a, s')$
    - Sometimes just $R(s)$ or $R(s')$
  - A start state (or distribution)
  - Maybe a terminal state

- MDPs: non-deterministic search problems
  - Reinforcement learning: MDPs where we don’t know the transition or reward functions
What is Markov about MDPs?

- Andrey Markov (1856-1922)
- “Markov” generally means that given the present state, the future and the past are independent
- For Markov decision processes, “Markov” means:

\[
P(S_{t+1} = s'|S_t = s_t, A_t = a_t, S_{t-1} = s_{t-1}, A_{t-1}, \ldots S_0 = s_0) = P(S_{t+1} = s'|S_t = s_t, A_t = a_t)
\]
Solving MDPs

- In deterministic single-agent search problems, want an optimal plan, or sequence of actions, from start to a goal.
- In an MDP, we want an optimal policy $\pi^*$: $S \rightarrow A$:
  - A policy $\pi$ gives an action for each state.
  - An optimal policy maximizes expected utility if followed.
  - Defines a reflex agent.

![Diagram](image)

Optimal policy when $R(s, a, s') = -0.03$ for all non-terminals $s$. 
Example Optimal Policies

\[ R(s) = -0.01 \]  
\[ R(s) = -0.03 \]  
\[ R(s) = -0.4 \]  
\[ R(s) = -2.0 \]
Another Example: Racing Car

- A robot car wants to travel far, quickly
- Three states: Cool, Warm, Overheated
- Two actions: Slow, Fast
- Going faster gets double reward

Optimal Policy:
- cool -> fast
- warm -> slow
Racing Car Search Tree
Example: High-Low

- Three card types: 2, 3, 4
- Infinite deck, twice as many 2’s
- Start with 3 showing
- After each card, you say “high” or “low”
- New card is flipped
- If you’re right, you win the points shown on the new card
- Ties are no-ops
- If you’re wrong, game ends

Differences from expectimax problems:
- #1: get rewards as you go
- #2: you might play forever!
High-Low as an MDP

- States: 2, 3, 4, done
- Actions: High, Low
- Model: \( T(s, a, s') \):
  - \( P(s'=4 \mid 4, \text{Low}) = 1/4 \)
  - \( P(s'=3 \mid 4, \text{Low}) = 1/4 \)
  - \( P(s'=2 \mid 4, \text{Low}) = 1/2 \)
  - \( P(s' = \text{done} \mid 4, \text{Low}) = 0 \)
  - \( P(s'=4 \mid 4, \text{High}) = 1/4 \)
  - \( P(s'=3 \mid 4, \text{High}) = 0 \)
  - \( P(s'=2 \mid 4, \text{High}) = 0 \)
  - \( P(s' = \text{done} \mid 4, \text{High}) = 3/4 \)
  - ...
- Rewards: \( R(s, a, s') \):
  - Number shown on \( s' \) if \( s \neq s' \)
  - 0 otherwise
- Start: 3
Search Tree: High-Low

3

Low  High

3, Low  3, High

T = 0.5, R = 2  T = 0.25, R = 3  T = 0, R = 4  T = 0.25, R = 0

2

High  Low  High  Low  High  Low
MDP Search Trees

- Each MDP state gives an expectimax-like search tree

\[ \text{(s, a)} \text{ is a q-state} \]

\[ \text{(s, a, s')} \text{ called a transition} \]

\[ T(s, a, s') = P(s'|s, a) \]

\[ R(s, a, s') \]
Utilities of Sequences

- What preference should an agent have over reward sequences?
  - More or less:
    - [1, 2, 2] or [2, 3, 4]
  - Now or later:
    - [0, 0, 1] or [1, 0, 0]
Utilities of Sequences

- In order to formalize optimality of a policy, need to understand utilities of sequences of rewards
- Typically consider stationary preferences:

\[ [r, r_0, r_1, r_2, \ldots] \succ [r, r_0', r_1', r_2', \ldots] \]

\[ \iff \]

\[ [r_0, r_1, r_2, \ldots] \succ [r_0', r_1', r_2', \ldots] \]

- Theorem: only two ways to define stationary utilities
  - Additive utility:
    \[ U([r_0, r_1, r_2, \ldots]) = r_0 + r_1 + r_2 + \cdots \]
  - Discounted utility:
    \[ U([r_0, r_1, r_2, \ldots]) = r_0 + \gamma r_1 + \gamma^2 r_2 \cdots \]
Infinite Utilities?! 

- Problem: infinite state sequences have infinite rewards

- Solutions:
  - Finite horizon:
    - Terminate episodes after a fixed $T$ steps (e.g. life)
    - Gives nonstationary policies ($\pi$ depends on time left)
  - Absorbing state: guarantee that for every policy, a terminal state will eventually be reached (like “done” for High-Low)
  - Discounting: for $0 < \gamma < 1$

$$U([r_0, \ldots r_\infty]) = \sum_{t=0}^{\infty} \gamma^t r_t \leq R_{\text{max}}/(1 - \gamma)$$

- Smaller $\gamma$ means smaller “horizon” – shorter term focus
Discounting

- It is reasonable to maximize the sum of rewards.
- It also makes sense to prefer rewards now to rewards later.
- One solution: value of rewards decay exponentially.

\[
\begin{align*}
\text{Worth now} & = 1 \\
\text{Worth in one step} & = \gamma \\
\text{Worth in two step} & = \gamma^2
\end{align*}
\]
**Discounting**

\[ U([r_0, \ldots r_\infty]) = \sum_{t=0}^{\infty} \gamma^t r_t \leq R_{\text{max}}/(1 - \gamma) \]

- Typically discount rewards by \( \gamma < 1 \) each time step
  - Sooner rewards have higher utility than later rewards
  - Also helps the algorithms converge
Recap: Defining MDPs

- **Markov decision processes:**
  - States $S$
  - Start state $s_0$
  - Actions $A$
  - Transitions $P(s'|s,a)$ (or $T(s,a,s')$)
  - Rewards $R(s,a,s')$ (and discount $\gamma$)

- **MDP quantities so far:**
  - Policy = Choice of action for each state
  - Utility (or return) = sum of discounted rewards
Define the value of a state $s$:

$$V^*(s) = \text{expected utility starting in } s \text{ and acting optimally}$$

Define the value of a q-state $(s,a)$:

$$Q^*(s,a) = \text{expected utility starting in } s, \text{ taking action } a \text{ and thereafter acting optimally}$$

Define the optimal policy:

$$\pi^*(s) = \text{optimal action from state } s$$
The Bellman Equations

Definition of “optimal utility” leads to a simple one-step lookahead relationship amongst optimal utility values:

Formally:

\[ V^*(s) = \max_a Q^*(s, a) \]

\[ Q^*(s, a) = \sum_{s'} T(s, a, s') \left[ R(s, a, s') + \gamma V^*(s') \right] \]

\[ V^*(s) = \max_a \sum_{s'} T(s, a, s') \left[ R(s, a, s') + \gamma V^*(s') \right] \]
Why Not Search Trees?

- Why not solve with expectimax?

- Problems:
  - This tree is usually infinite (why?)
  - Same states appear over and over (why?)
  - We would search once per state (why?)

- Idea: Value iteration
  - Compute optimal values for all states all at once using successive approximations
  - Will be a bottom-up dynamic program similar in cost to memoization
  - Do all planning offline, no replanning needed!
Racing Car Search Tree

- We’re doing way too much work with expectimax!

- Problem: States are repeated
  - Idea: Only compute needed quantities once

- Problem: Tree goes on forever
  - Idea: Do a depth-limited computation, but with increasing depths until change is small
  - Note: deep parts of the tree eventually don’t matter if $\gamma < 1$
Value Estimates

- Calculate estimates $V_k^*(s)$
  - The optimal value considering only next $k$ time steps ($k$ rewards)
  - As $k \to \infty$, it approaches the optimal value

- Why:
  - If discounting, distant rewards become negligible
  - If terminal states reachable from everywhere, fraction of episodes not ending becomes negligible
  - Otherwise, can get infinite expected utility and then this approach actually won’t work
Computing time limited values

$V_4(\cdot) \quad V_4(\cdot) \quad V_4(\cdot)$

$V_3(\cdot) \quad V_3(\cdot) \quad V_3(\cdot)$

$V_2(\cdot) \quad V_2(\cdot) \quad V_2(\cdot)$

$V_1(\cdot) \quad V_1(\cdot) \quad V_1(\cdot)$

$V_0(\cdot) \quad V_0(\cdot) \quad V_0(\cdot)$
Value Iteration

- **Idea:**
  - Start with $V_0^*(s) = 0$, which we know is right (why?)
  - Given $V_i^*$, calculate the values for all states for depth $i+1$:

  $$V_{i+1}(s) \leftarrow \max_a \sum_{s'} T(s, a, s') \left[ R(s, a, s') + \gamma V_i(s') \right]$$

  - This is called a value update or Bellman update
  - Repeat until convergence

- **Theorem:** will converge to unique optimal values
  - Basic idea: approximations get refined towards optimal values
  - Policy may converge long before values do
Example: Value Iteration
Example: Bellman Updates

\[ V_{i+1}(s) = \max_a \sum_{s'} T(s, a, s') \left[ R(s, a, s') + \gamma V_i(s') \right] = \max_a Q_{i+1}(s, a) \]

\[ Q_1(\langle 3, 3 \rangle, \text{right}) = \sum_{s'} T(\langle 3, 3 \rangle, \text{right}, s') \left[ R(\langle 3, 3 \rangle, \text{right}, s') + \gamma V_i(s') \right] \]

\[ = 0.8 \times [0.0 + 0.9 \times 1.0] + 0.1 \times [0.0 + 0.9 \times 0.0] + 0.1 \times [0.0 + 0.9 \times 0.0] \]
Example: Value Iteration

- Information propagates outward from terminal states and eventually all states have correct value estimates.
Example of Value iteration

\[ V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s, a, s') \left[ R(s, a, s') + \gamma V_k(s') \right] \]
Example of Value iteration

\[
V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s, a, s') \left[ R(s, a, s') + \gamma V_k(s') \right]
\]
Convergence

- Define the max-norm: \( \|U\| = \max_s |U(s)| \)

- Theorem: For any two approximations \( U \) and \( V \)
  \[ \|U^{t+1} - V^{t+1}\| \leq \gamma \|U^t - V^t\| \]
  - i.e. any distinct approximations must get closer to each other, so, in particular, any approximation must get closer to the true \( U \) and value iteration converges to a unique, stable, optimal solution

- Theorem:
  \[ \|U^{t+1} - U^t\| < \epsilon, \Rightarrow \|U^{t+1} - U\| < 2\epsilon\gamma/(1 - \gamma) \]
  - i.e. once the change in our approximation is small, it must also be close to correct
Value Iteration Complexity

- **Problem size:**
  - \(|A|\) actions and \(|S|\) states

- **Each Iteration**
  - Computation: \(O(|A| \cdot |S|^2)\)
  - Space: \(O(|S|)\)

- **Num of iterations**
  - Can be exponential in the discount factor \(\gamma\)
Practice: Computing Actions

Which action should we chose from state s:

- Given optimal values $Q$?
  
  $$\arg \max_a Q^*(s, a)$$

- Given optimal values $V$?
  
  $$\arg \max_a \sum_{s'} T(s, a, s') [R(s, a, s') + \gamma V^*(s')]$$

- Lesson: actions are easier to select from $Q$’s!
Aside: Q-Value Iteration

- **Value iteration**: find successive approx optimal values
  - Start with $V_0^*(s) = 0$
  - Given $V_i^*$, calculate the values for all states for depth $i+1$:
    \[
    V_{i+1}(s) \leftarrow \max_a \sum_{s'} T(s, a, s') \left[ R(s, a, s') + \gamma V_i(s') \right]
    \]

- But Q-values are more useful!
  - Start with $Q_0^*(s, a) = 0$
  - Given $Q_i^*$, calculate the q-values for all q-states for depth $i+1$:
    \[
    Q_{i+1}(s, a) \leftarrow \sum_{s'} T(s, a, s') \left[ R(s, a, s') + \gamma \max_{a'} Q_i(s', a') \right]
    \]
Another basic operation: compute the utility of a state \( s \) under a fix (general non-optimal) policy

Define the utility of a state \( s \), under a fixed policy \( \pi \):

\[
V^\pi(s) = \text{expected total discounted rewards (return) starting in } s \text{ and following } \pi
\]

Recursive relation (one-step look-ahead / Bellman equation):

\[
V^\pi(s) = \sum_{s'} T(s, \pi(s), s')[R(s, \pi(s), s') + \gamma V^\pi(s')]
\]
Policy Evaluation

- How do we calculate the V’s for a fixed policy?
  - **Idea one:** modify Bellman updates
    
    \[
    V_0^{\pi}(s) = 0
    \]
    
    \[
    V_{i+1}^{\pi}(s) \leftarrow \sum_{s'} T(s, \pi(s), s')[R(s, \pi(s), s') + \gamma V_i^{\pi}(s')] \]
  
- **Idea two:** it’s just a linear system, solve with Matlab (or whatever)
Policy Iteration

- Problem with value iteration:
  - Considering all actions each iteration is slow: takes $|A|$ times longer than policy evaluation
  - But policy doesn’t change each iteration, time wasted

- Alternative to value iteration:
  - **Step 1: Policy evaluation**: calculate utilities for a fixed policy (not optimal utilities!) until convergence (fast)
  - **Step 2: Policy improvement**: update policy using one-step lookahead with resulting converged (but not optimal!) utilities (slow but infrequent)
  - Repeat steps until policy converges
Policy Iteration

- **Policy evaluation**: with fixed current policy $\pi$, find values with simplified Bellman updates
  - Iterate until values converge

  $$V_{i+1}^{\pi_k}(s) \leftarrow \sum_{s'} T(s, \pi_k(s), s') \left[R(s, \pi_k(s), s') + \gamma V_i^{\pi_k}(s')\right]$$

  - Note: could also solve value equations with other techniques

- **Policy improvement**: with fixed utilities, find the best action according to one-step look-ahead

  $$\pi_{k+1}(s) = \arg\max_a \sum_{s'} T(s, a, s') \left[R(s, a, s') + \gamma V^{\pi_k}(s')\right]$$
Policy Iteration Complexity

- **Problem size:**
  - $|A|$ actions and $|S|$ states

- **Each Iteration**
  - Computation: $O(|S|^3 + |A| \cdot |S|^2)$
  - Space: $O(|S|)$

- **Num of iterations**
  - Unknown, but can be faster in practice
  - Convergence is guaranteed
Comparison

- **In value iteration:**
  - Every pass (or “backup”) updates both utilities (explicitly, based on current utilities) and policy (possibly implicitly, based on current policy)

- **In policy iteration:**
  - Several passes to update utilities with frozen policy
  - Occasional passes to update policies

- **Hybrid approaches (asynchronous policy iteration):**
  - Any sequences of partial updates to either policy entries or utilities will converge if every state is visited infinitely often