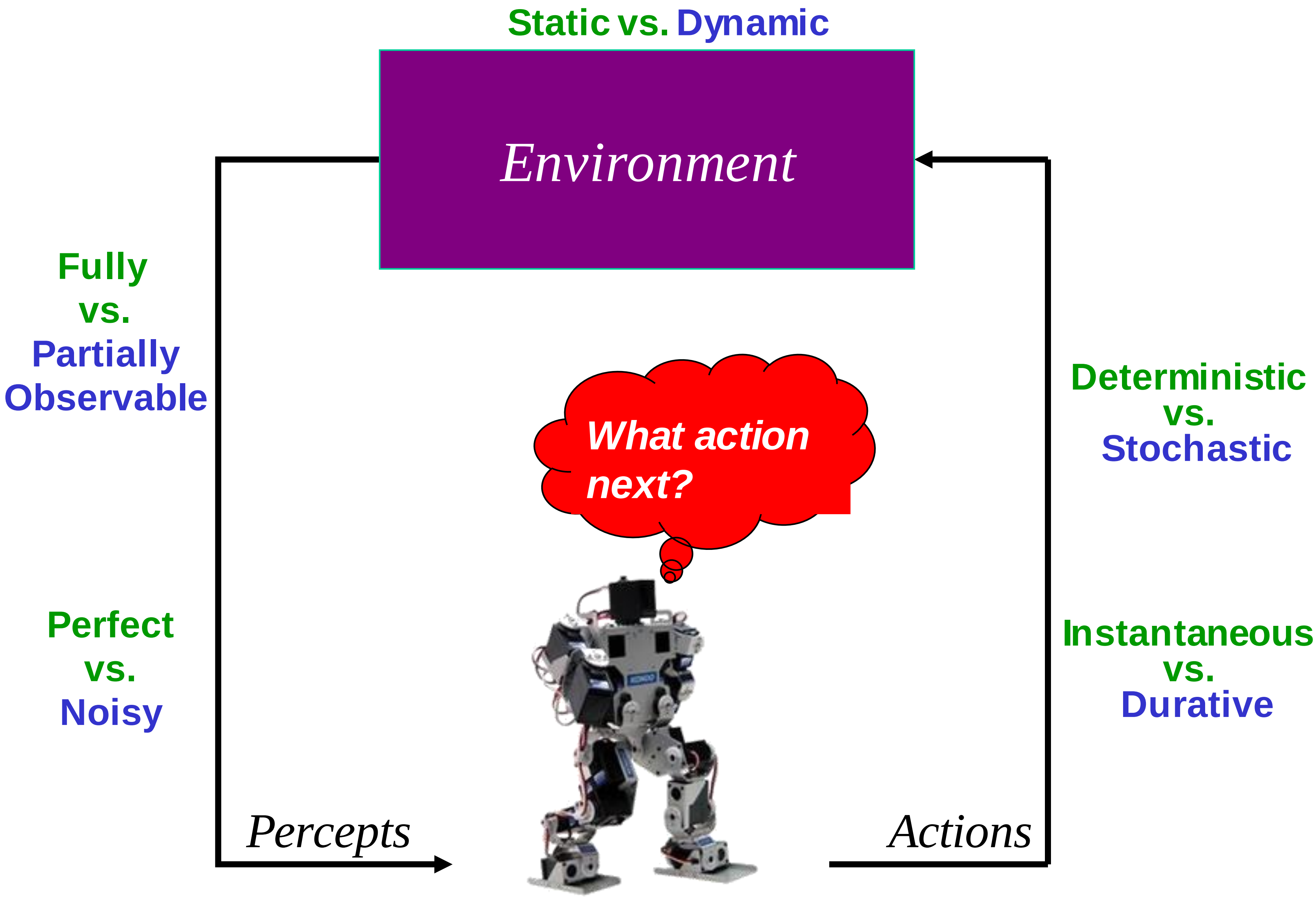
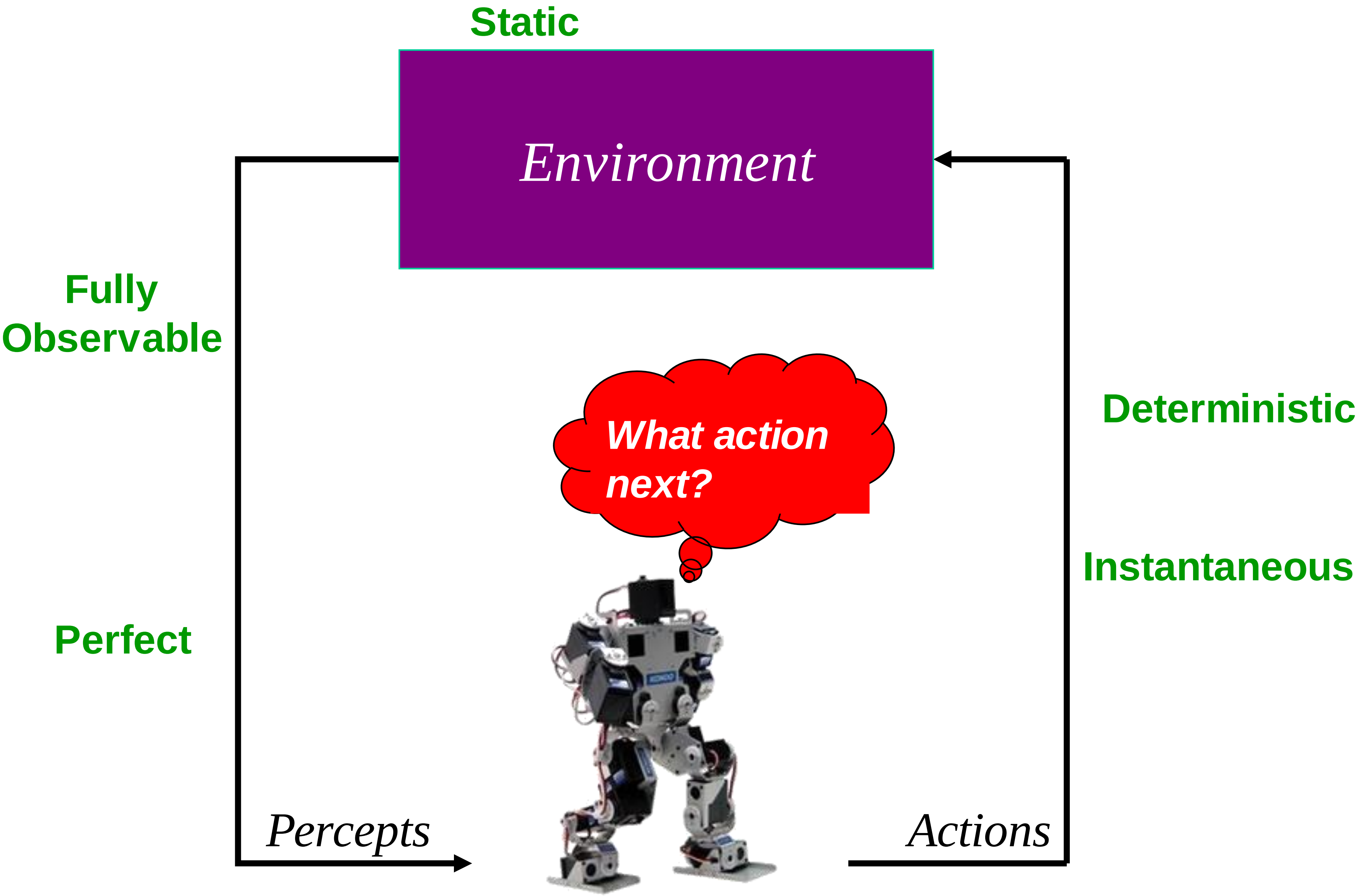
Markov Decision Processes
Chapter 17

Mausam

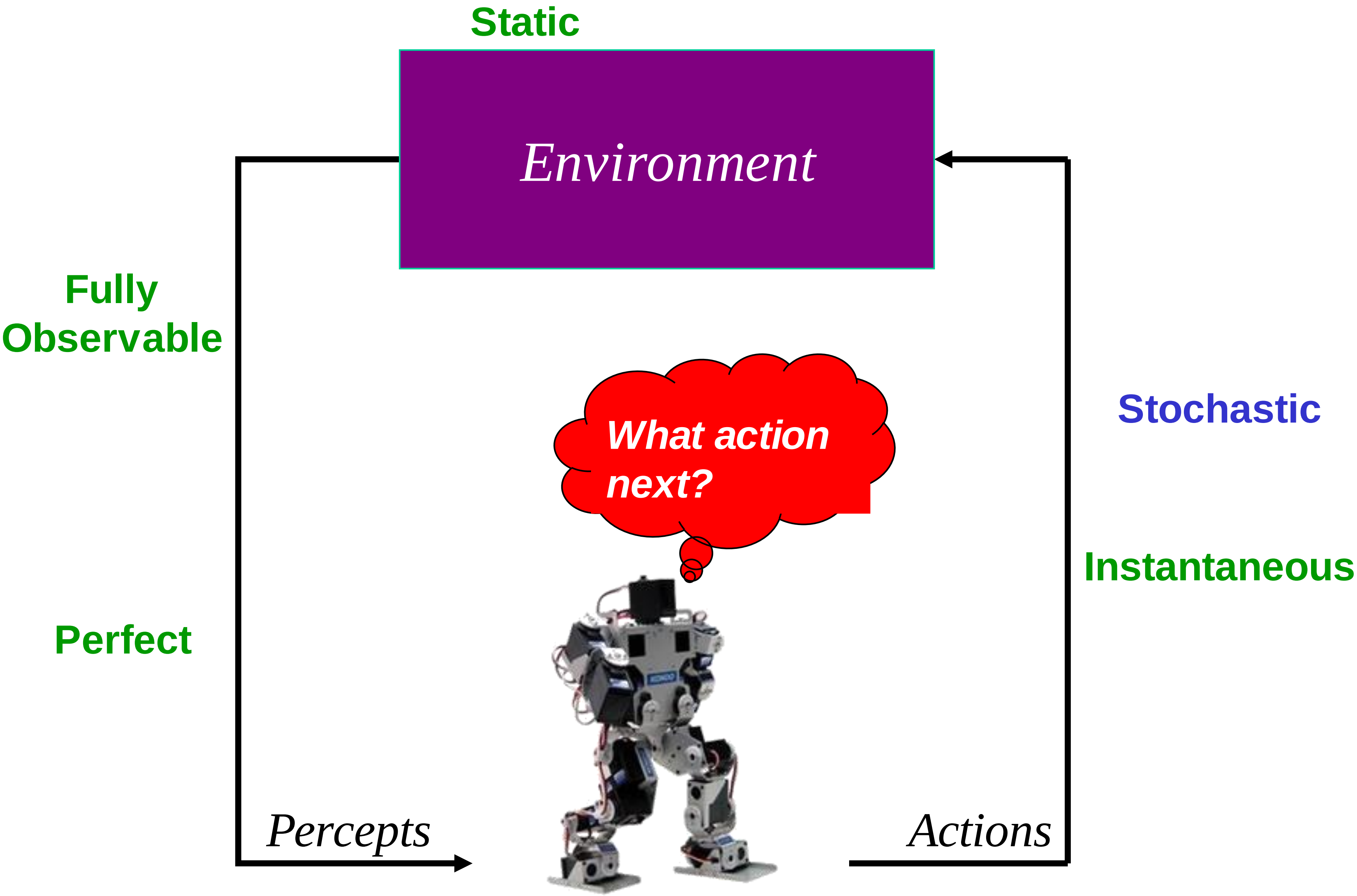
Planning Agent



Classical Planning



Stochastic Planning: MDPs



MDP vs. Decision Theory

- Decision theory - episodic
- MDP -- sequential

Markov Decision Process (MDP)

- $\mathcal{S}$: A set of states
- $\mathcal{A}$: A set of actions
- $\mathcal{P}r(s'|s,a)$: transition model
- $\mathcal{C}(s,a,s')$: cost model
- $\mathcal{G}$: set of goals
- s_0: start state
- γ: discount factor
- $\mathcal{R}(s,a,s')$: reward model

factored

Factored MDP

absorbing/
non-absorbing

Objective of an MDP

- Find a policy $\pi: \mathcal{S} \rightarrow \mathcal{A}$

- which optimizes
  - minimizes discounted expected cost to reach a goal
  - maximizes or expected reward
  - maximizes undiscount. expected (reward-cost)

- given a _____ horizon
  - finite
  - infinite
  - indefinite

- assuming full observability

Role of Discount Factor (γ)

- Keep the total reward/total cost finite
  - useful for infinite horizon problems
- Intuition (economics):
  - Money today is worth more than money tomorrow.
- Total reward: $r_1 + \gamma r_2 + \gamma^2 r_3 + \ldots$
- Total cost: $c_1 + \gamma c_2 + \gamma^2 c_3 + \ldots$

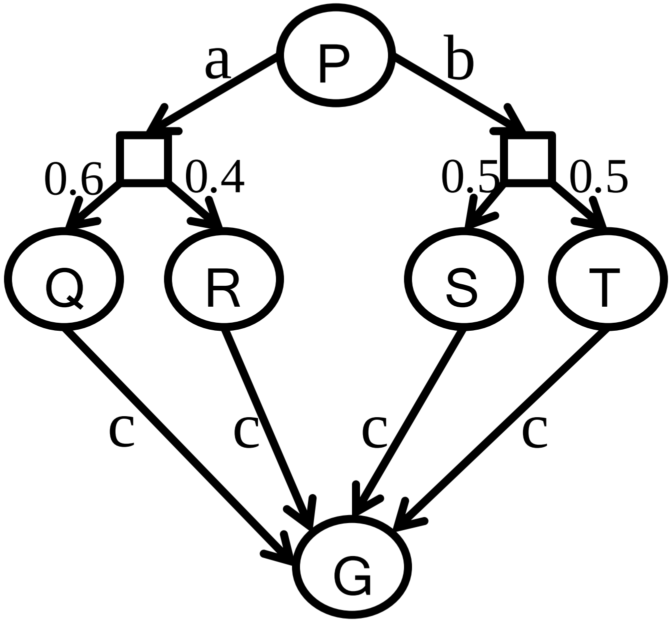
Examples of MDPs

- Goal-directed, Indefinite Horizon, Cost Minimization MDP
  - <$\mathcal{S}$, $\mathcal{A}$, $\mathcal{P}$r, $\mathcal{C}$, $\mathcal{G}$, s_0>
  - Most often studied in planning, graph theory communities
- Infinite Horizon, Discounted Reward Maximization MDP
  - <$\mathcal{S}$, $\mathcal{A}$, $\mathcal{P}$r, $\mathcal{R}$, γ>
  - Most often studied in machine learning, economics, operations research communities

most popular

- Oversubscription Planning: Non absorbing goals, Reward Max. MDP
  - <$\mathcal{S}$, $\mathcal{A}$, $\mathcal{P}$r, $\mathcal{G}$, $\mathcal{R}$, s_0>
  - Relatively recent model

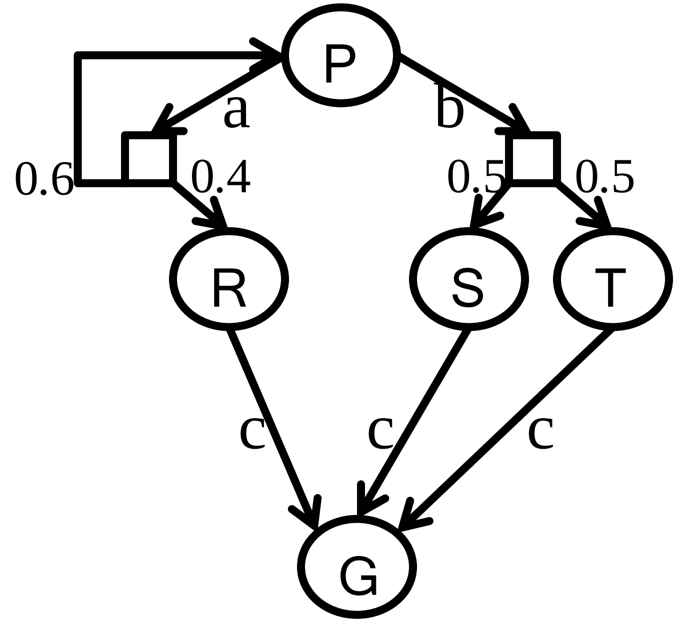
AND/OR Acyclic Graphs vs. MDPs



C(a) = 5, C(b) = 10, C(c) =1

Expectimin works

- V(Q/R/S/T) = 1
- V(P) = 6 – action a



Expectimin doesn't work

- infinite loop

- V(R/S/T) = 1
- Q(P,b) = 11
- Q(P,a) = ????
- suppose I decide to take a in P
- Q(P,a) = 5+ 0.4*1 + 0.6Q(P,a)
- ➔ = 13.5

Bellman Equations for MDP_1

- <$\mathcal{S}$, $\mathcal{A}$, $\mathcal{P}$r, $\mathcal{C}$, $\mathcal{G}$, s_0>
- Define J*(s) {optimal cost} as the minimum expected cost to reach a goal from this state.
- J* should satisfy the following equation:

$$\boxed{\begin{aligned} J^*(s) &= 0 \;\; if \;\; s \in \mathcal{G} \\ J^*(s) &= \min_{a \in Ap(s)} \sum_{s' \in \mathcal{S}} \mathcal{P}r(s'|s,a)\left[\mathcal{C}(s,a,s') + J^*(s')\right] \end{aligned}}$$

Bellman Equations for MDP_2

- <$\mathcal{S}$, $\mathcal{A}$, $\mathcal{P}$r, $\mathcal{R}$, s_0, γ>
- Define **V*(s)** {optimal **value**} as the **maximum** expected **discounted reward** from this state.
- V* should satisfy the following equation:

$$\boxed{V^*(s) \;=\; \max_{a \in Ap(s)} \sum_{s' \in \mathcal{S}} \mathcal{P}r(s'|s,a)\left[\mathcal{R}(s,a,s') + \gamma V^*(s')\right]}$$

Bellman Backup (MDP_2)

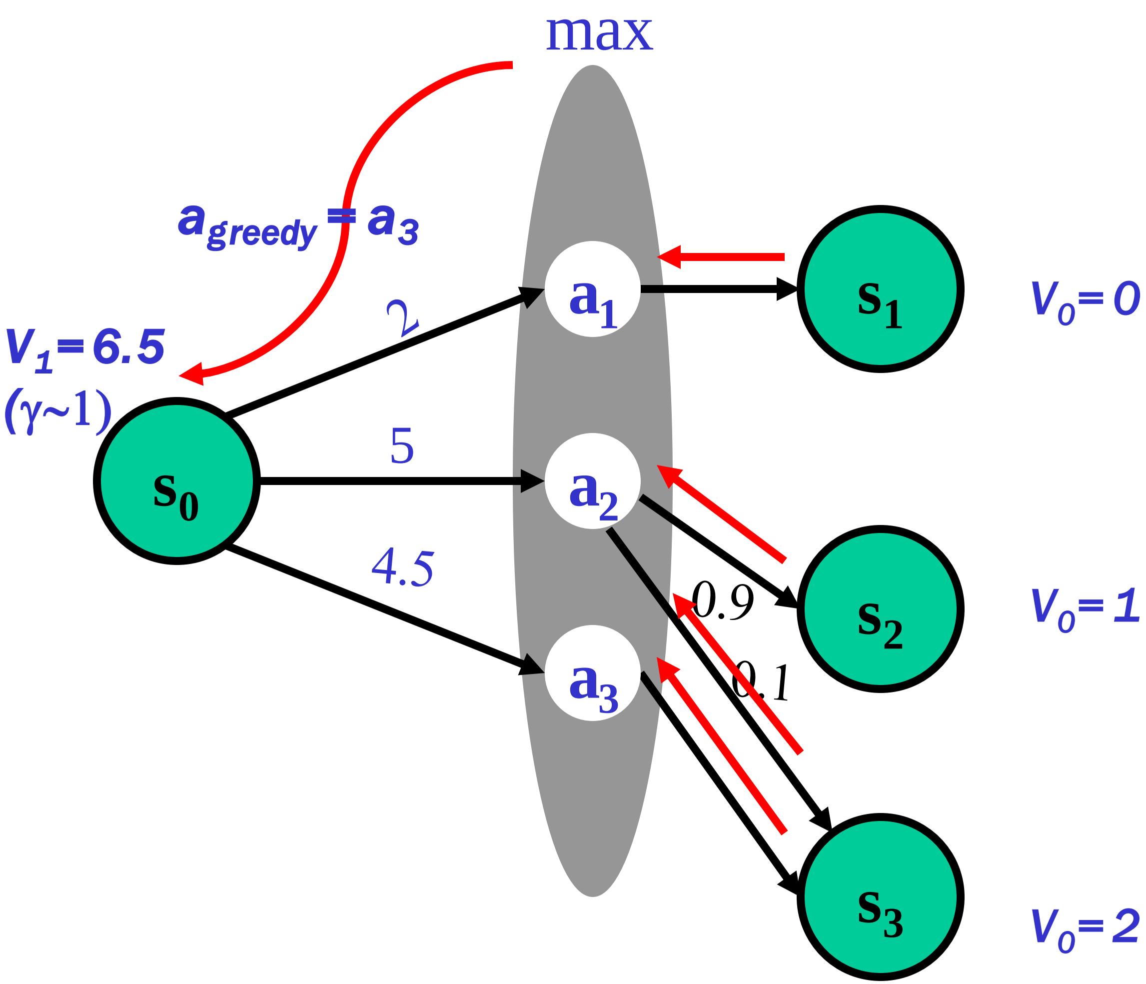
- Given an estimate of V* function (say V_n)
- Backup V_n function at state s
  - calculate a new estimate (V_{n+1}) :

$$\boxed{\begin{aligned} Q_{n+1}(s,a) &= \sum_{s' \in \mathcal{S}} Pr(s'|s,a) \left[\mathcal{R}(s,a,s') + \gamma V_n(s')\right] \\ V_{n+1}(s) &= \max_{a \in Ap(s)} \left[Q_{n+1}(s,a)\right] \end{aligned}}$$

- $Q_{n+1}(s,a)$: value/cost of the strategy:
  - execute action a in s, execute π_n subsequently
  - $\pi_n = \mathrm{argmax}_{a \in Ap(s)} Q_n(s,a)$

Bellman Backup

$Q_1(s,a_1) = 2 + 0\,\gamma$
$Q_1(s,a_2) = 5 + \gamma\, 0.9 \times 1$
$\qquad\qquad + \gamma\, 0.1 \times 2$
$Q_1(s,a_3) = 4.5 + 2\,\gamma$



Value iteration [Bellman'57]

- assign an arbitrary assignment of V_0 to each state.
- repeat
  - for all states s
    - compute $V_{n+1}(s)$ by Bellman backup at s.
- until $\max_s |V_{n+1}(s) - V_n(s)| < \varepsilon$

Iteration n+1

Residual(s)

ε-convergence

Comments

- Decision-theoretic Algorithm
- Dynamic Programming
- Fixed Point Computation
- Probabilistic version of Bellman-Ford Algorithm
  - for shortest path computation
  - MDP_1 : Stochastic Shortest Path Problem

- Time Complexity
  - one iteration: $O(|\mathcal{S}|^2|\mathcal{A}|)$
  - number of iterations: poly($|\mathcal{S}|$, $|\mathcal{A}|$, $1/(1-\gamma)$)
- Space Complexity: $O(|\mathcal{S}|)$
- Factored MDPs = Planning under uncertainty
  - exponential space, exponential time

Convergence Properties

- $V_n \rightarrow V^*$ in the limit as $n \rightarrow \infty$
- ε-convergence: V_n function is within ε of V^*
- Optimality: current policy is within $2\varepsilon\gamma/(1-\gamma)$ of optimal

- Monotonicity
  - $V_0 \leq_p V^* \Rightarrow V_n \leq_p V^*$ (V_n monotonic from below)
  - $V_0 \geq_p V^* \Rightarrow V_n \geq_p V^*$ (V_n monotonic from above)
  - otherwise V_n non-monotonic

Policy Computation

$$\begin{aligned}\pi^*(s) &= \operatorname*{argmax}_{a \in Ap(s)} Q^*(s,a) \\ &= \operatorname*{argmax}_{a \in Ap(s)} \sum_{s' \in \mathcal{S}} \mathcal{P}r(s'|s,a) \left[\mathcal{R}(s,a,s') + \gamma V^*(s')\right]\end{aligned}$$

Policy Evaluation

$$V_\pi(s) = \sum_{s' \in \mathcal{S}} \mathcal{P}r(s'|s,\pi(s)) \left[\mathcal{R}(s,\pi(s),s') + \gamma V_\pi(s')\right]$$

A system of linear equations in $|\mathcal{S}|$ variables.

Changing the Search Space

- Value Iteration
  - Search in value space
  - Compute the resulting policy

- Policy Iteration
  - Search in policy space
  - Compute the resulting value

Policy iteration [Howard'60]

- assign an arbitrary assignment of π_0 to each state.

- repeat
  - Policy Evaluation: compute V_{n+1}: the evaluation of π_n
  - Policy Improvement: for all states s
    - compute $\pi_{n+1}(s)$: $\text{argmax}_{a\in Ap(s)} Q_{n+1}(s,a)$
- until $\pi_{n+1} = \pi_n$

costly: $O(n^3)$

Modified Policy Iteration → approximate by value iteration using fixed policy

Advantage

- searching in a finite (policy) space as opposed to uncountably infinite (value) space ⇒ convergence faster.
- all other properties follow!

Modified Policy iteration

- assign an arbitrary assignment of π_0 to each state.

- repeat
  - Policy Evaluation: compute V_{n+1} the *approx.* evaluation of π_n
  - Policy Improvement: for all states s
    - compute $\pi_{n+1}(s)$: $\text{argmax}_{a \in Ap(s)} Q_{n+1}(s,a)$
- until $\pi_{n+1} = \pi_n$

Advantage

- probably the most competitive synchronous dynamic programming algorithm.

Applications

- Stochastic Games
- Robotics: navigation, helicopter manuevers…
- Finance: options, investments
- Communication Networks
- Medicine: Radiation planning for cancer
- Controlling workflows
- Optimize bidding decisions in auctions
- Traffic flow optimization
- Aircraft queueing for landing; airline meal provisioning
- Optimizing software on mobiles
- Forest firefighting
- …

Extensions

- Heuristic Search + Dynamic Programming
  - AO*, LAO*, RTDP, …
- Factored MDPs
  - add planning graph style heuristics
  - use goal regression to generalize better
- Hierarchical MDPs
  - hierarchy of sub-tasks, actions to scale better
- Reinforcement Learning
  - learning the probability and rewards
  - acting while learning – connections to psychology
- Partially Observable Markov Decision Processes
  - noisy sensors; partially observable environment
  - popular in robotics

Summary: Decision Making

- Classical planning
  - sequential decision making in deterministic world
  - domain independent heuristic generation
- Decision theory
  - one step decision making under uncertainty
  - value of information
  - utility theory
- Markov Decision Process
  - sequential decision making under uncertainty