14. Parametric surfaces

Reading

Recommended:

- Watt, 2.2 and 6.4.
- Bartels, Beatty, and Barsky. *An Introduction to Splines for use in Computer Graphics and Geometric Modeling,* 1987.

Mathematical surface representations

- Explicit $z=f(x,y)$ (a.k.a., a "height field")
  - what if the curve isn't a function, like a sphere?

- Implicit $g(x,y,z) = 0$

- Parametric $(x(u,v),y(u,v),z(u,v))$
  - For the sphere:

    $x(u,v) = r \cos 2\pi v \sin \pi u$

    $y(u,v) = r \sin 2\pi v \sin \pi u$

    $z(u,v) = r \cos \pi u$

As with curves, we'll focus on parametric surfaces.

Surfaces of revolution

Idea: rotate a 2D **profile curve** around an axis.

What kinds of shapes can you model this way?

Constructing surfaces of revolution

Given: A curve $C(u)$ in the *xy*-plane:

$$C(u) = \begin{bmatrix} c_x(u) \\ c_y(u) \\ 0 \\ 1 \end{bmatrix}$$

Let $R_x(\theta)$ be a rotation about the *x*-axis.

Find: A surface $S(u,v)$ which is $C(u)$ rotated about the *x*-axis.

Solution:

General sweep surfaces

The **surface of revolution** is a special case of a **swept surface**.

Idea: Trace out surface $S(u,v)$ by moving a **profile curve** $C(u)$ along a **trajectory curve** $T(v)$.

More specifically:

- Suppose that $C(u)$ lies in an (x_c,y_c) coordinate system with origin O_c.
- For every point along $T(v)$, lay $C(u)$ so that O_c coincides with $T(v)$.

Orientation

The big issue:

- How to orient $C(u)$ as it moves along $T(v)$?

Here are two options:

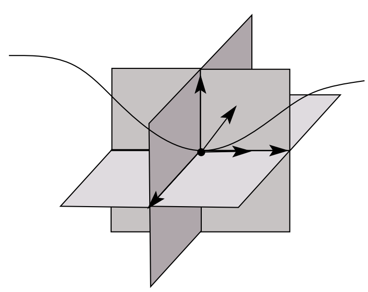
1. **Fixed** (or **static**): Just translate O_c along $T(v)$.

2. Moving. Use the **Frenet frame** of $T(v)$.

- Allows smoothly varying orientation.
- Permits surfaces of revolution, for example.

Frenet frames

Motivation: Given a curve $T(v)$, we want to attach a smoothly varying coordinate system.



To get a 3D coordinate system, we need 3 independent direction vectors.

$$\hat{t}(v) = normalize(T'(v))$$
$$\hat{b}(v) = normalize(T'(v) \times T''(v))$$
$$\hat{n}(v) = \hat{b}(v) \times \hat{t}(v)$$

As we move along $T(v)$, the Frenet frame (t,b,n) varies smoothly.

Frenet swept surfaces

Orient the profile curve $C(u)$ using the Frenet frame of the trajectory $T(v)$:

- Put $C(u)$ in the **normal plane** *nb*.
- Place O_c on $T(v)$.
- Align x_c for $C(u)$ with b.
- Align y_c for $C(u)$ with $-n$.

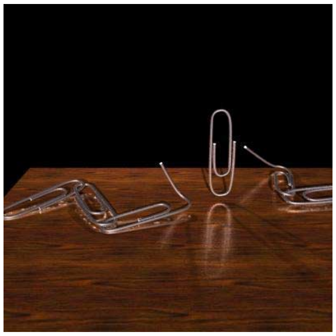
If $T(v)$ is a circle, you get a surface of revolution exactly!

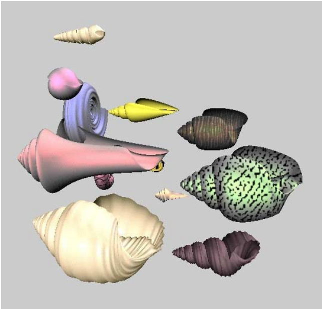
What happens at inflection points, I.e., where curvature goes to zero?

Variations

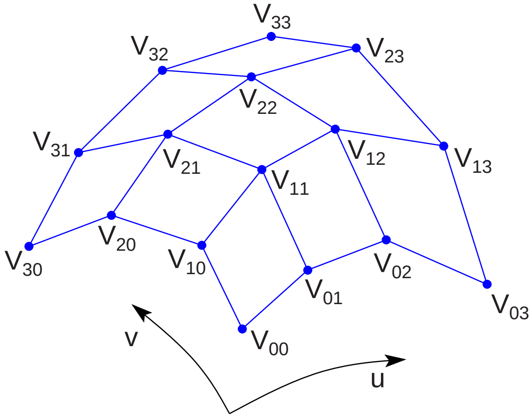
Several variations are possible:

- Scale $C(u)$ as it moves, possibly using length of $T(v)$ as a scale factor.
- Morph $C(u)$ into some other curve $C'(u)$ as it moves along $T(v)$.
- …





Tensor product Bézier surfaces

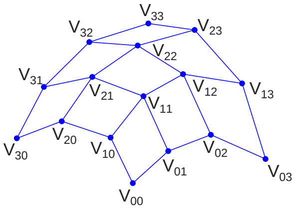


Given a grid of control points V_{ij}, forming a **control net**, contruct a surface $S(u,v)$ by:

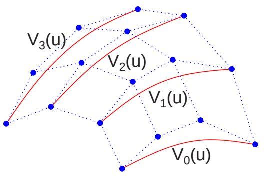
- treating rows of V as control points for curves $V_0(u),\ldots, V_n(u)$.
- treating $V_0(u),\ldots, V_n(u)$ as control points for a curve parameterized by v.

Tensor product surfaces, cont.

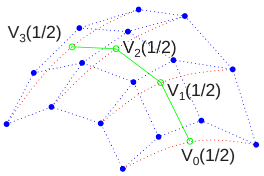
Let's walk through the steps:



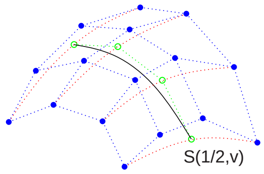
Control net



Control curves in *u*



Control polygon at *u*=1/2



Curve at *S*(1/2,*v*)

Which control points are interpolated by the surface?

Matrix form

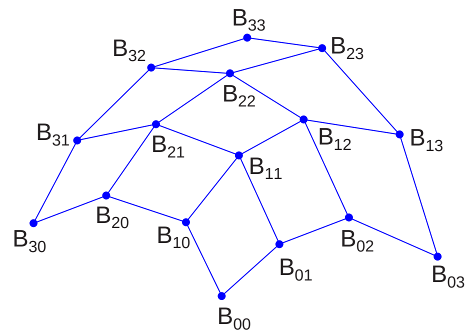
Tensor product surfaces can be written out explicitly:

$$S(u,v) = \sum_{i=0}^{n}\sum_{j=0}^{n} V_{ij} B_i^n(u) B_j^n(v)$$

$$= \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} M_{Bézier} \; V \; M_{Bézier}^T \begin{bmatrix} v^3 \\ v^2 \\ v \\ 1 \end{bmatrix}$$

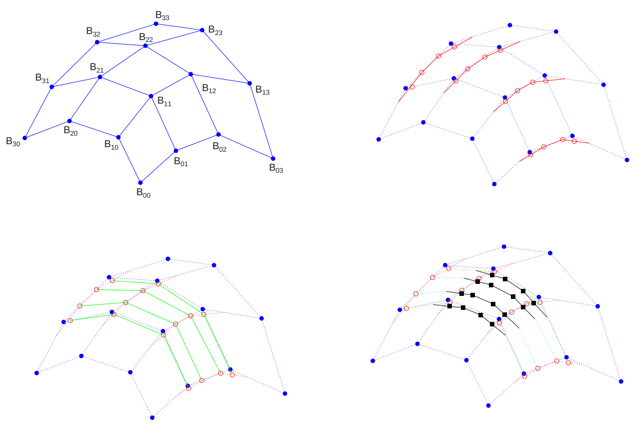
Tensor product B-spline surfaces

As with spline curves, we can piece together a sequence of Bézier surfaces to make a spline surface. If we enforce C2 continuity and local control, we get B-spline curves:



- treat rows of *B* as control points to generate Bézier control points in *u*.
- treat Bézier control points in *u* as B-spline control points in *v*.
- treat B-spline control points in *v* to generate Bézier control points in *u*.

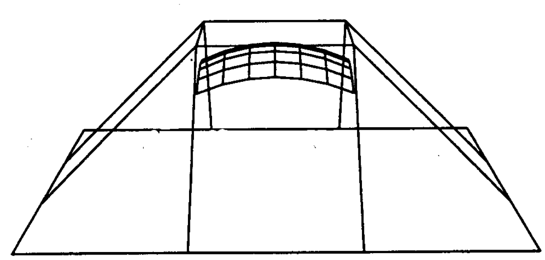
Tensor product B-splines, cont.



Which B-spline control points are interpolated by the surface?

Tensor product B-splines, cont.

Another example:



Trimmed NURBS surfaces

Uniform B-spline surfaces are a special case of NURBS surfaces.

Sometimes, we want to have control over which parts of a NURBS surface get drawn.

For example:

We can do this by **trimming** the *u-v* domain.

- Define a closed curve in the *u-v* domain (a **trim curve**)
- Do not draw the surface points inside of this curve.

It's really hard to maintain continuity in these regions, especially while animating.

Summary

What to take home:

- How to construct swept surfaces from a profile and trajectory curve:
  - with a fixed frame
  - with a Frenet frame
- How to construct tensor product Bézier surfaces
- How to construct tensor product B-spline surfaces