Week 6 study sheet: Curves

Problem 1 (28 points) A Bézier curve of degree n, which (for the purposes of this problem) we'll donate by $Q^n(u)$, can be defined in terms of the locations of its $n+1$ control points $\{V_0, \ldots, V_n\}$:

$$Q^n(u) \quad = \quad \sum_{i=0}^{n} V_i \left(\begin{array}{c} n \\ i \end{array} \right) u^i \, (1-u)^{n-i}$$

a) (4 points) Use de Casteljau's algorithm to find the (approximate) position of the Bézier curves $Q^1(u)$ and $Q^2(u)$ defined by the two control polygons below at $u = 1/3$:

Problem 1 (continued)

True or false:

b) (2 points) Every Bézier curve $Q^1(u)$ is a line segment (assuming no repeated control points).

c) (2 points) Every Bézier curve $Q^2(u)$ lies in a plane.

d) (3 points) The tangent of $Q^1(u)$ at $u = 0$ is always 0.

e) (3 points) The curvature of $Q^2(u)$ at $u = 0$ is always 0.

f) (7 points) For every Bézier curve $Q^1(u)$ there exists a curve $Q^2(u)$ such that $Q^1(u) = Q^2(u)$ for all u. (For full credit, provide a proof of your answer for this part.)

Problem 1 (continued)

Since a Bézier curve $Q^n(u)$ can be expressed as a linear combination of control points

$$B_0(u)V_0 \ + \ B_1(u)V_1 \ + \ \cdots \ + \ B_n(u)V_n,$$

we can associate an interval $[u_{\min}, u_{\min}]$ with each control point V_i, defined as the region of the parameter domain u for which $B_i(u)$ is larger than any of the other "basis functions" $B_0(u), \ldots, B_n(u)$. We'll call this interval the *dominant region* of a control point V_i.

For example, for Bézier curves of degree 1, the dominant region of V_0 is $[0, 1/2]$, and the dominant region of V_1 is $[1/2, 1]$.

g) (7 points) What is the dominant region for each of the three control points V_0, V_1, V_2 of a Bézier curve of degree 2? (For full credit, provide a proof of your answer.)

Problem 2 (20 points) More complex curves can be designed by piecing together different Bézier curves to make mathematical "splines." Two popular splines are the B-spline and the Catmull-Rom spline. If $\{B_0, B_1, B_2, B_3\}$ and $\{C_0, C_1, C_2, C_3\}$ are cubic B-spline and Catmull-Rom spline control points, respectively, then the corresponding Bézier control points $\{V_0, V_1, V_2, V_3\}$ can be constructed by the following identity:

$$\begin{bmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 1 & 4 & 1 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 1 & 4 & 1 \end{bmatrix}\begin{bmatrix} B_0 \\ B_1 \\ B_2 \\ B_3 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 0 & 6 & 0 & 0 \\ -1 & 6 & 1 & 0 \\ 0 & 1 & 6 & -1 \\ 0 & 0 & 6 & 0 \end{bmatrix}\begin{bmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \end{bmatrix}$$

True or false:

a) (2 points) B-splines and Catmull-Rom splines both have C^2 continuity.

b) (2 points) Neither B-splines nor Catmull-Rom splines interpolate their control points.

c) (2 points) B-splines and Catmull-Rom splines both lie inside the convex hull of their control points.

d) (2 points) B-splines and Catmull-Rom splines both provide local control.

Problem 2 (continued)

e) (6 points) The points $\{B_0, B_1, B_2, B_3\}$ below are control points for a cubic B-spline. Construct, as carefully as you can on the diagram below, the Bézier control points $\{V_0, V_1, V_2, V_3\}$ corresponding to the same curve.

f) (6 points) The points $\{C_0, C_1, C_2, C_3\}$ below are control points for a cubic Catmull-Rom spline. Construct, as carefully as you can on the diagram below, the Bézier control points $\{V_0, V_1, V_2, V_3\}$ corresponding to the same curve.