Affine Transformations

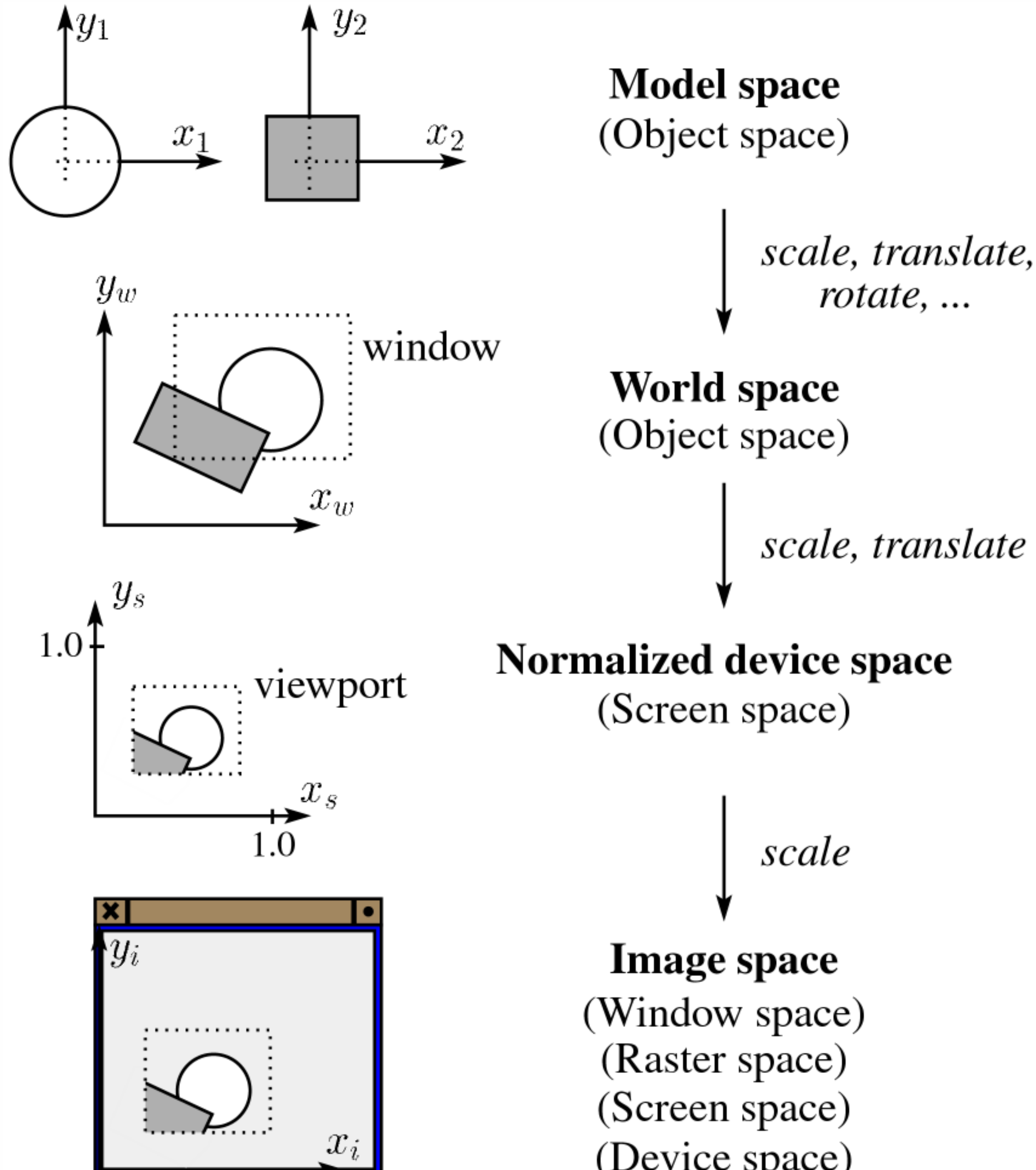
Reading

- Foley et al., Chapter 5.6 and Chapter 6

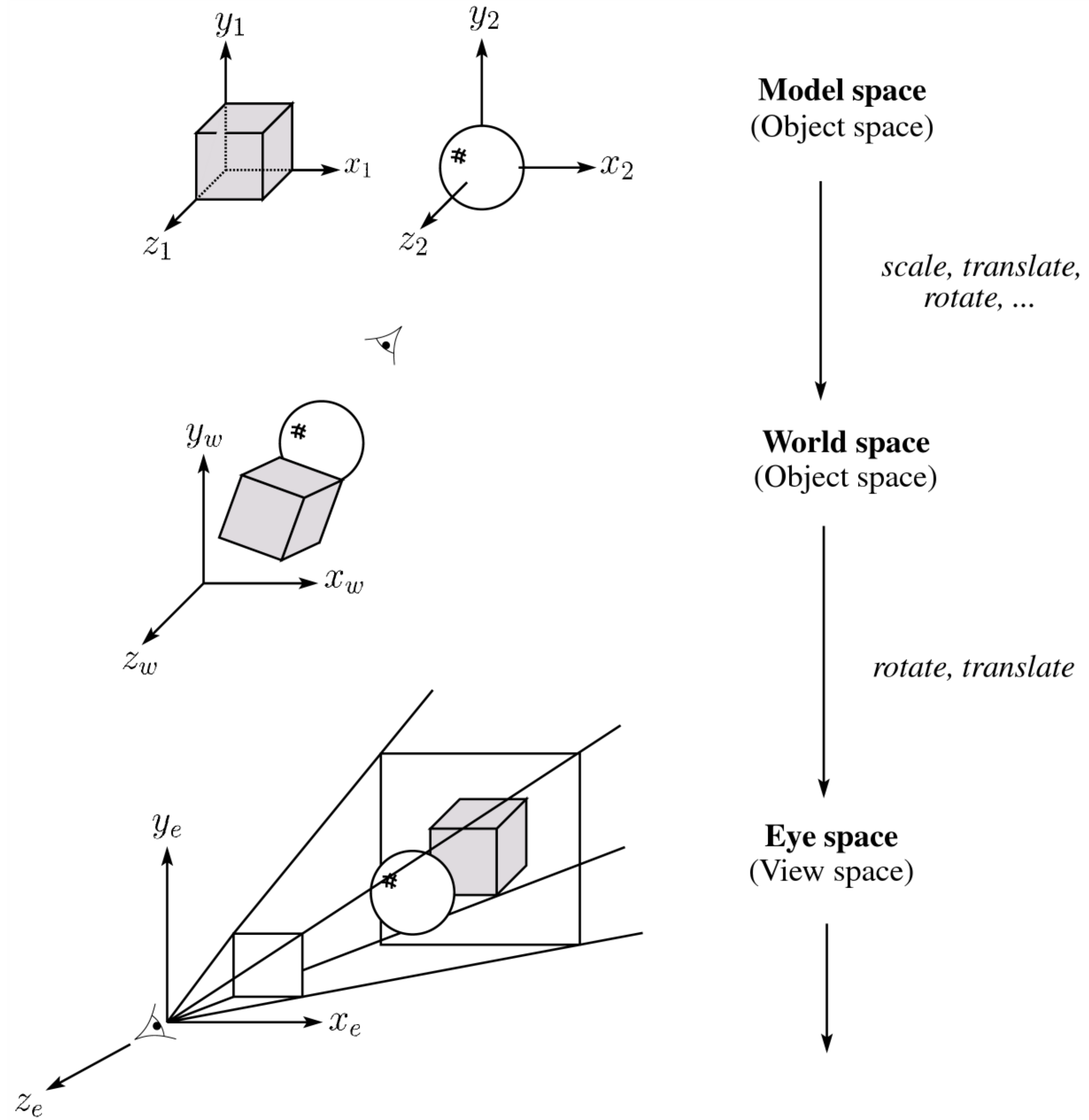
Supplemental

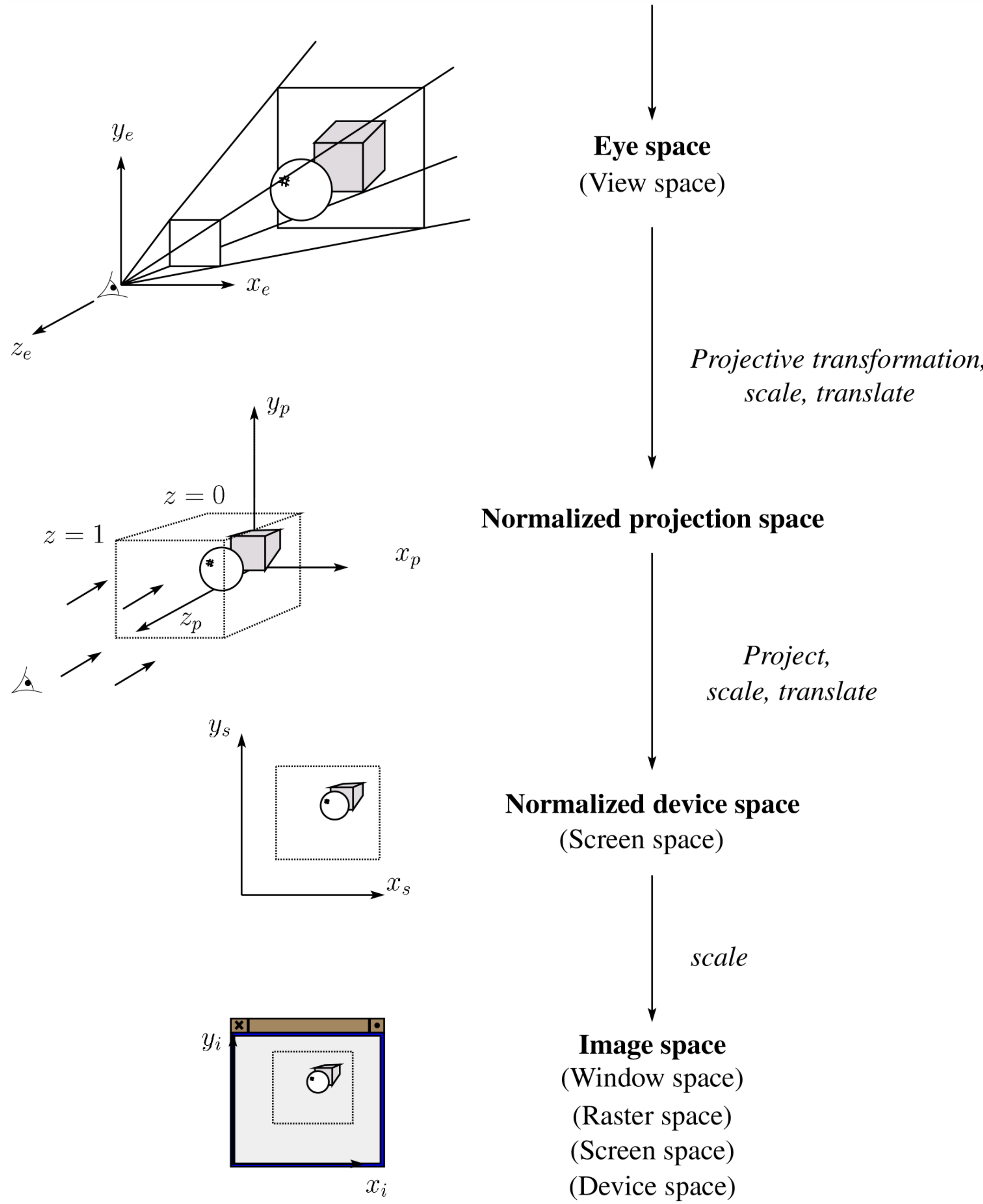
- David F. Rogers and J. Alan Adams, *Mathematical Elements for Computer Graphics, Second edition*

2D geometry Pipeline



3D Geometry Pipeline

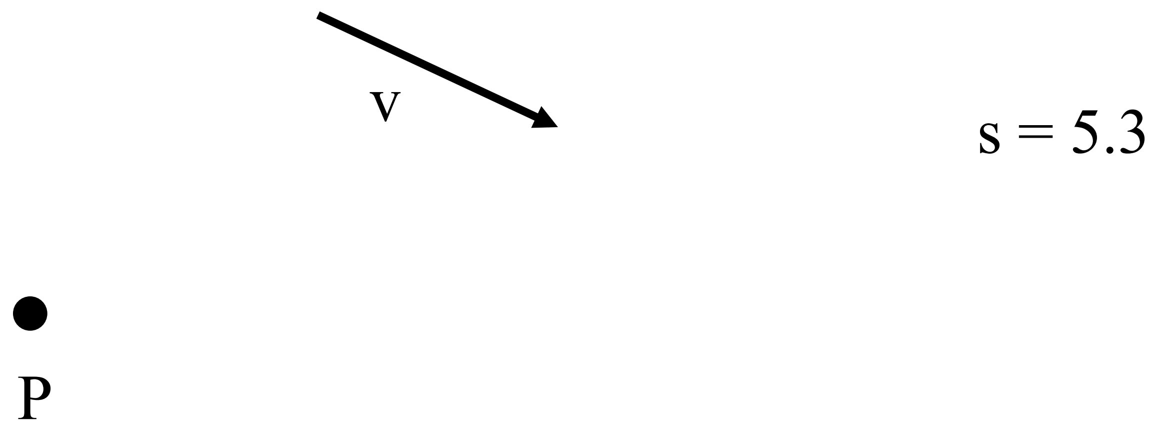




y_e
x_e
z_e
Eye space
(View space)
Projective transformation,
scale, translate
y_p
$z = 0$
$z = 1$
x_p
z_p
Normalized projection space
Project,
scale, translate
y_s
x_s
Normalized device space
(Screen space)
scale
y_i
x_i
Image space
(Window space)
(Raster space)
(Screen space)
(Device space)

Affine Geometry

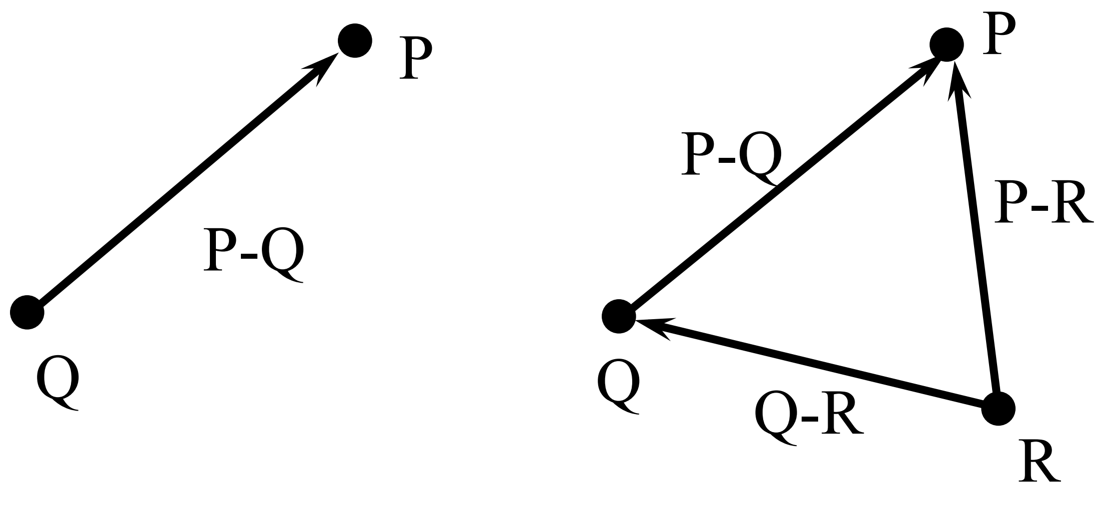
- Points: location in 3D space
- Vectors: quantity with a direction and magnitude, but no fixed position
- Scalar: a real number



Affine Spaces

Affine space consists of points and vectors related by a set of axioms:

- Difference of two points is a vector:
- Head-to-tail rule for vector addition:



Affine Operations

Legal affine operations:

$$\text{vector} + \text{vector} \rightarrow \text{vector}$$

$$\text{scalar} \cdot \text{vector} \rightarrow \text{vector}$$

$$\text{point} - \text{point} \rightarrow \text{vector}$$

$$\text{point} + \text{vector} \rightarrow \text{point}$$

… example of an "illegal" operation:

$$\text{point} + \text{point} \rightarrow \text{nonsense}$$

Useful combination of affine operations:

$$P(\alpha) = P_0 + \alpha \mathbf{v}$$

What is it?

Affine Combination

Affine combination of two points:

$$Q = \alpha_1 Q_1 + \alpha_2 Q_2$$

where $\alpha_1 + \alpha_2 = 1$ is defined to be the point

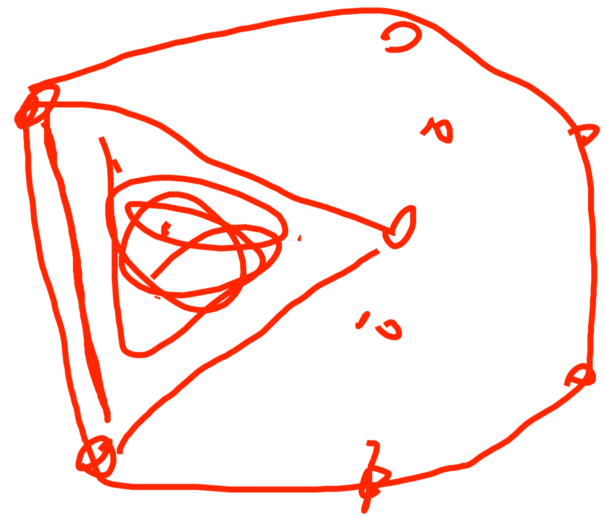
$$Q = Q_1 + a_2 (Q_2 - Q_1)$$

We can generalize affine combination to multiple points:

$$Q = \alpha_1 Q_1 + \alpha_2 Q_2 + \mathrm{L} + \alpha_n Q_n$$

where

$$\sum \alpha_i = 1$$



Affine Frame

A frame can be defined as a set of vectors and a point:

$$\left(\mathbf{v}_1, \ldots, \mathbf{v}_n, O\right)$$

Where $\mathbf{v}_1, \ldots, \mathbf{v}_n$ form a basis and O is a point in space.

Any point P can be written as

$$P = p_1\mathbf{v}_1 + \ldots + p_n\mathbf{v}_n + O$$

And any vector as:

$$\mathbf{u} = u_1\mathbf{v}_1 + \ldots + u_n\mathbf{v}_n$$

Matrix representation of points and vectors

Coordinate axiom: $0 \cdot P = \mathbf{0}$

$$1 \cdot P = P$$

So every point in the frame $\mathcal{F} = (\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n, \mathcal{O})$ can be written as

$$P = p_1\mathbf{v}_1 + p_2\mathbf{v}_2 + \cdots + p_n\mathbf{v}_n + 1 \cdot \mathcal{O}$$

$$= \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n & \mathcal{O} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \\ 1 \end{bmatrix}$$

And every vector as

$$\mathbf{u} = u_1\mathbf{v}_1 + u_2\mathbf{v}_2 + \cdots + u_n\mathbf{v}_n + 0 \cdot \mathcal{O}$$

$$= \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n & \mathcal{O} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \\ 0 \end{bmatrix}$$

Changing frames

Given a point P in frame $\mathcal{F}$, what are the coordinates of P in frame $\mathcal{F}' = (\mathbf{v}_1', \mathbf{v}_2', \ldots, \mathbf{v}_n', \mathcal{O}')$

$$P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n & \mathcal{O} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1' & \mathbf{v}_2' & \cdots & \mathbf{v}_n' & \mathcal{O}' \end{bmatrix} \begin{bmatrix} p_1' \\ p_2' \\ \vdots \\ p_n' \\ 1 \end{bmatrix}$$

Since each element of $\mathcal{F}$ can be written in coordinates relative to $\mathcal{F}'$

$$\mathbf{v}_i = f_{i,1}\mathbf{v}_1' + \cdots + f_{i,n}\mathbf{v}_n'$$

$$\mathcal{O} = f_{n+1,1}\mathbf{v}_1' + \cdots + f_{n+1,n}\mathbf{v}_n' + \mathcal{O}'$$

Changing frames cont'd

Written in a matrix form

$$
\begin{bmatrix} \mathbf{v}'_1 & \mathbf{v}'_2 & \cdots & \mathbf{v}'_n & O' \end{bmatrix}
\begin{bmatrix} p'_1 \\ p'_2 \\ \vdots \\ p'_n \\ 1 \end{bmatrix}
=
\begin{bmatrix} \mathbf{v}'_1 & \mathbf{v}'_2 & \cdots & \mathbf{v}'_n & O' \end{bmatrix}
\begin{bmatrix} f_{1,1} & \cdots & f_{n,1} & f_{n+1,1} \\ \vdots & \ddots & & \vdots \\ f_{1,n} & & f_{n,n} & f_{n+1,n} \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \\ 1 \end{bmatrix}
$$

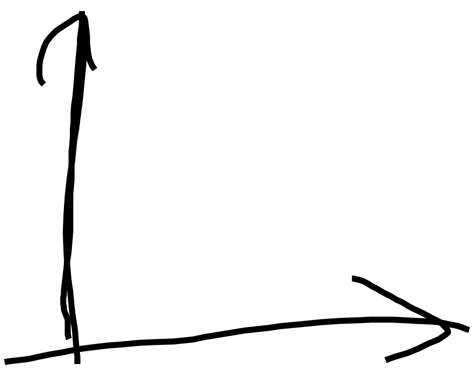
$$
\begin{bmatrix} p'_1 \\ p'_2 \\ \vdots \\ p'_n \\ 1 \end{bmatrix}
=
\begin{bmatrix} f_{1,1} & \cdots & f_{n,1} & f_{n+1,1} \\ \vdots & \ddots & & \vdots \\ f_{1,n} & & f_{n,n} & f_{n+1,n} \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \\ 1 \end{bmatrix}
= \mathbf{F}
\begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \\ 1 \end{bmatrix}
$$

Euclidean and Cartesian spaces

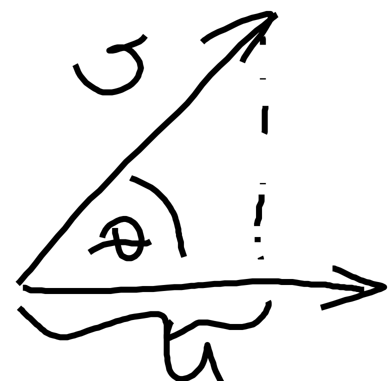
A Euclidean space is an affine space with an inner product:

$$\langle u, v \rangle = u \cdot v = u^T v$$

A Cartesian space is a Euclidean space with a standard orthonormal frame. In 3D: (**e**1, **e**2, **e**2, [illegible])



$$\mathbf{e}_i \cdot \mathbf{e}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$



$u \cdot v = \|u\| \cdot \|v\| \cos\theta$

Useful properties and operations in Cartesian spaces

Length: $|\mathbf{v}| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$

Distance between points: $|P - Q|$

Angle between vectors: $\cos^{-1}\left(\dfrac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| \cdot |\mathbf{v}|}\right)$

Perpendicular (orthogonal): $\mathbf{u} \cdot \mathbf{v} = 0$

Parallel: $\dfrac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| \cdot |\mathbf{v}|} = \pm 1$

Cross product (in 3D): $\mathbf{u} \times \mathbf{v} = \mathbf{w}$

Affine Transformations

$F : A \rightarrow B$ is an affine transformation if it preserves affine combinations:

$$F\left(\sum \alpha_i Q_i\right) = \sum \alpha_i F(Q_i)$$

Where $\sum \alpha_i = 1$. The same applies to vectors.

Affine coordiantes are preserved: $F\left(0 + \sum p_i \mathbf{v}_i\right) = F(0) + \sum p_i F(\mathbf{v}_i)$

Lines map to lines: $F(P_0 + \alpha \mathbf{v}) = F(P_0) + \alpha F(\mathbf{v})$

Paralelism is preserved: $F(Q_0 + \beta \mathbf{v}) = F(Q_0) + \beta F(\mathbf{v})$

Ratios are preserved: $Ratio\left(Q_1, Q, Q_2\right) = Ratio\left(F(Q_1), F(Q), F(Q_2)\right)$

2D Affine Transformations

P=[x,y,1]

P is a column vector

$$P' = \mathbf{M}P$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

P is a row vector

$$P' = P\mathbf{M}$$

$$\begin{bmatrix} x' & y' & 1 \end{bmatrix} = \begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} a & d & 0 \\ b & e & 0 \\ c & f & 1 \end{bmatrix}$$

Identity

Doesn't move points at all

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Translation

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$x' = x + c$$

$$y' = y + f$$

Scaling

Changing the diagonal elements performs scaling

$$\begin{bmatrix} a & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{aligned} x' &= ax \\ y' &= fy \end{aligned}$$

If $a=f$ scaling is uniform

What if $a,f<0$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Shearing

What about the off-diagonal elements?

The matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ d & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Gives

$$x' = x$$

$$y' = dx + y$$

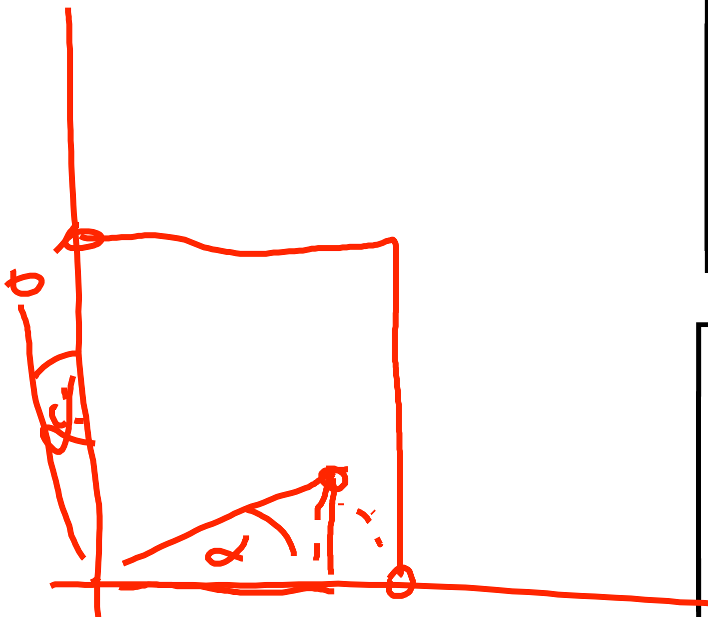
Effect on unit square

$$\begin{bmatrix} a & b & 0 \\ d & e & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & a & a+b & b \\ 0 & d & d+e & e \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

- M can be determined just by knowing how corners [1,0,1] and [0,1,1] are mapped
- a and e give x- and y-scaling
- b and d give x- and y-shearing

Rotation

- Rotation of points [1,0,1] and [0,1,1] by angle α around the origin:



$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} \cos(\alpha) \\ \sin(\alpha) \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} -\sin(\alpha) \\ \cos(\alpha) \\ 1 \end{bmatrix}$$

The Matrices

Identity (do nothing): $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Scale by s_x in the x and s_y in the y direction ($s_x < 0$ or $s_y < 0$ is reflection): $\begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Rotate by angle θ (in radians): $\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Shear by amount a in the x direction: $\begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Shear by amount b in the y direction: $\begin{bmatrix} 1 & 0 & 0 \\ b & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Translate by the vector (t_x, t_y): $\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$

Transformation Composition

Applying transformations **F** to point P and transformation **G** to the result

$$P' = \mathbf{F}P$$

$$P'' = \mathbf{G}P'$$

Combining two transformations

$$P'' = \mathbf{G}(\mathbf{F}P)$$

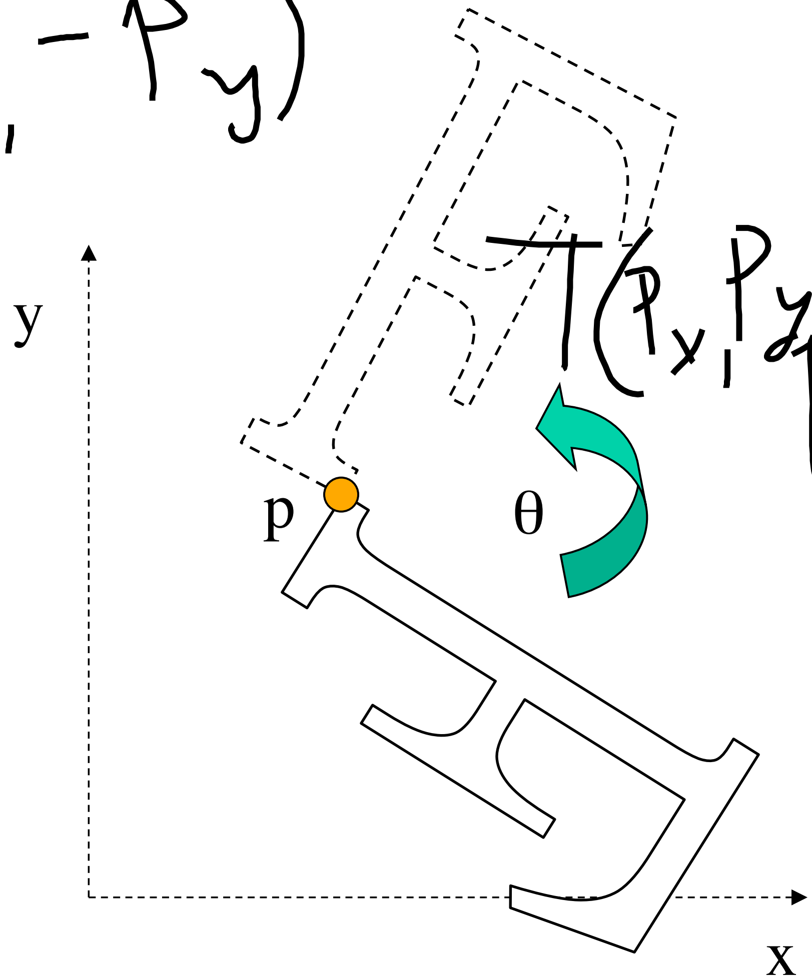
$$= (\mathbf{G}\mathbf{F})P$$

Let’s play a game

- Problems 2,3,4,14,17,18

Rotation around arbitrary point

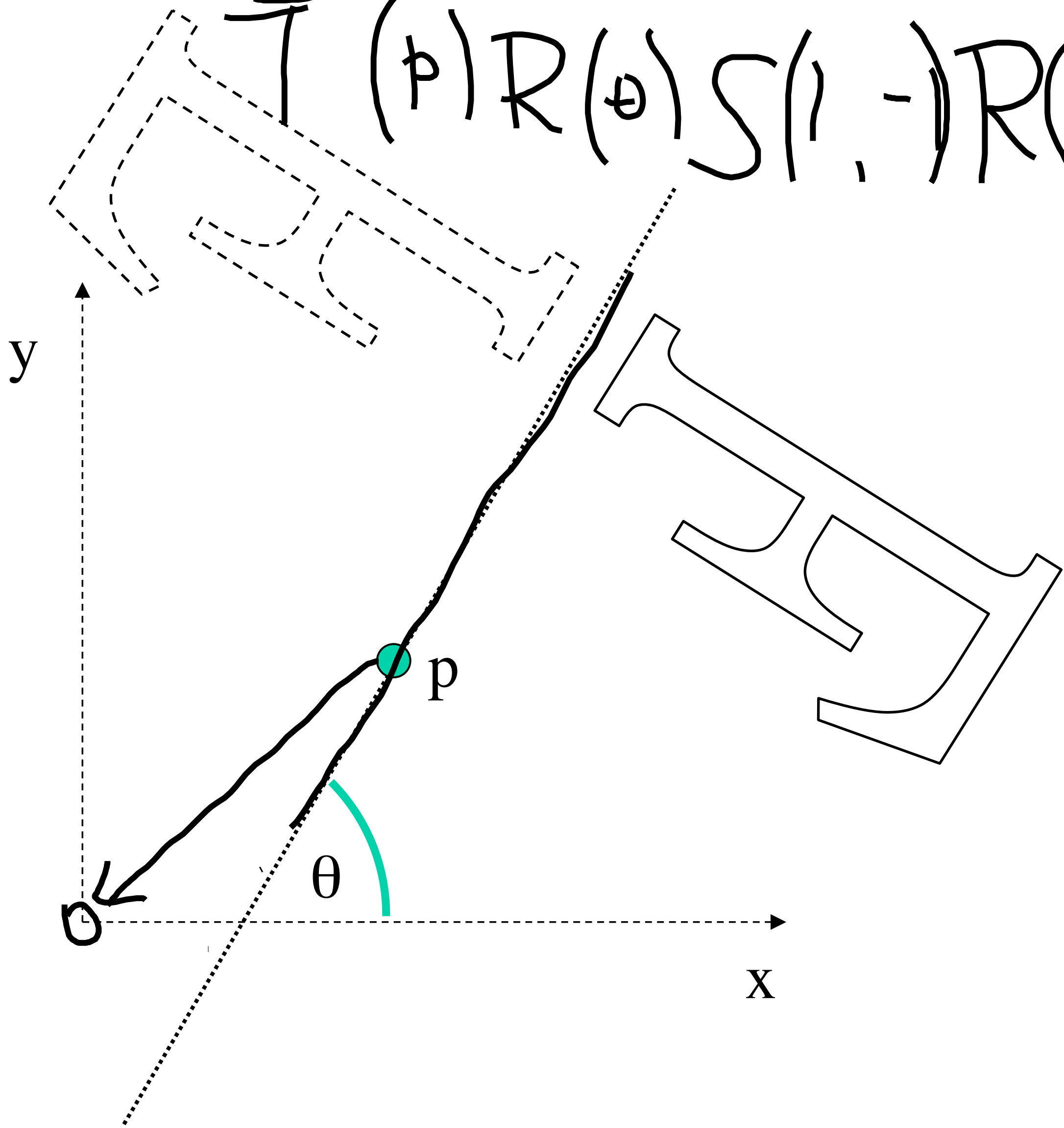
$T(-p_x, -p_y)$



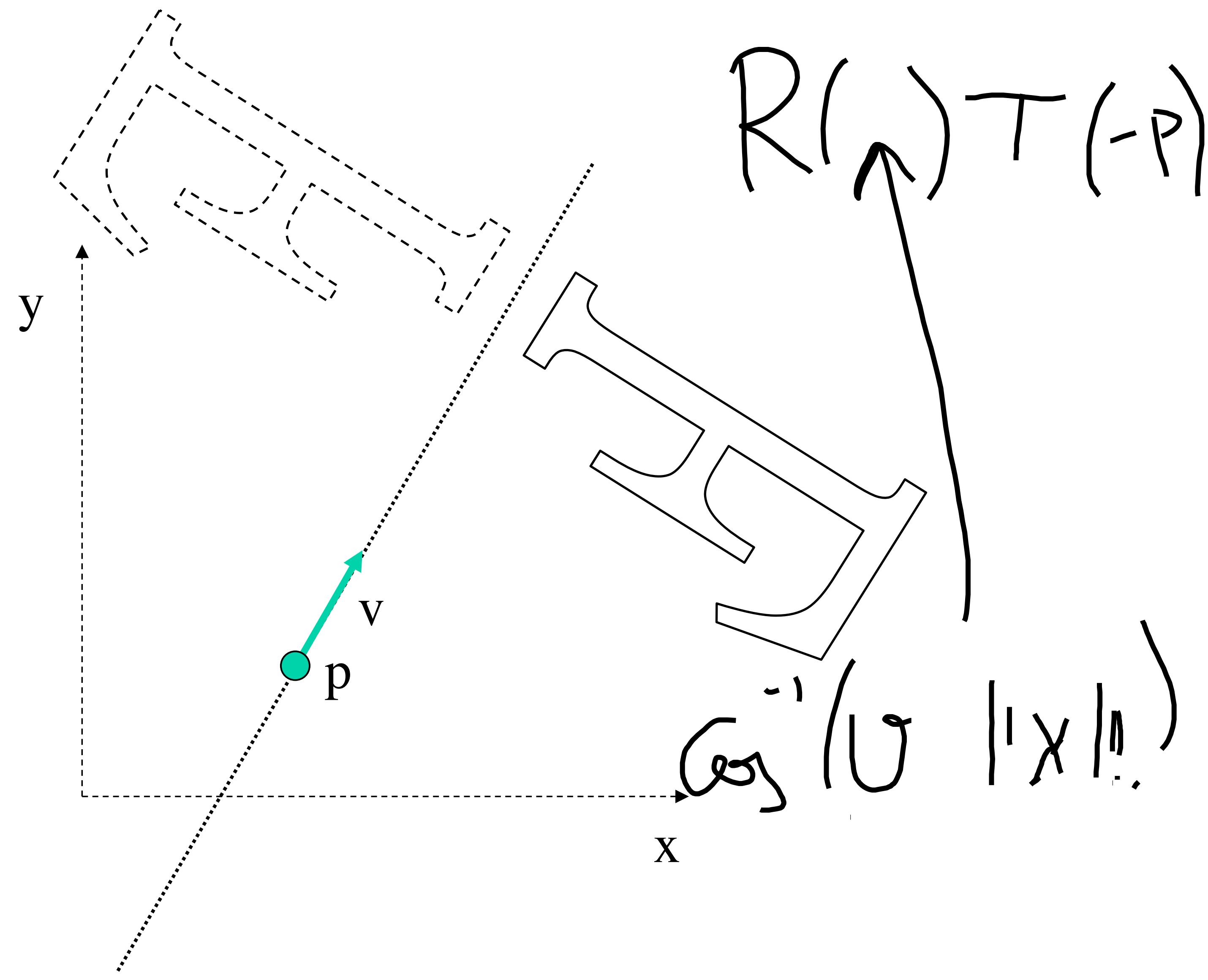
$T(p_x, p_y)\,R(\theta)\,T(-p_x, -p_y)$

Reflection around arbitrary axis

$T(p)R(\theta)S(1,-1)R(-\theta)T(-p)$



Reflection around arbitrary axis

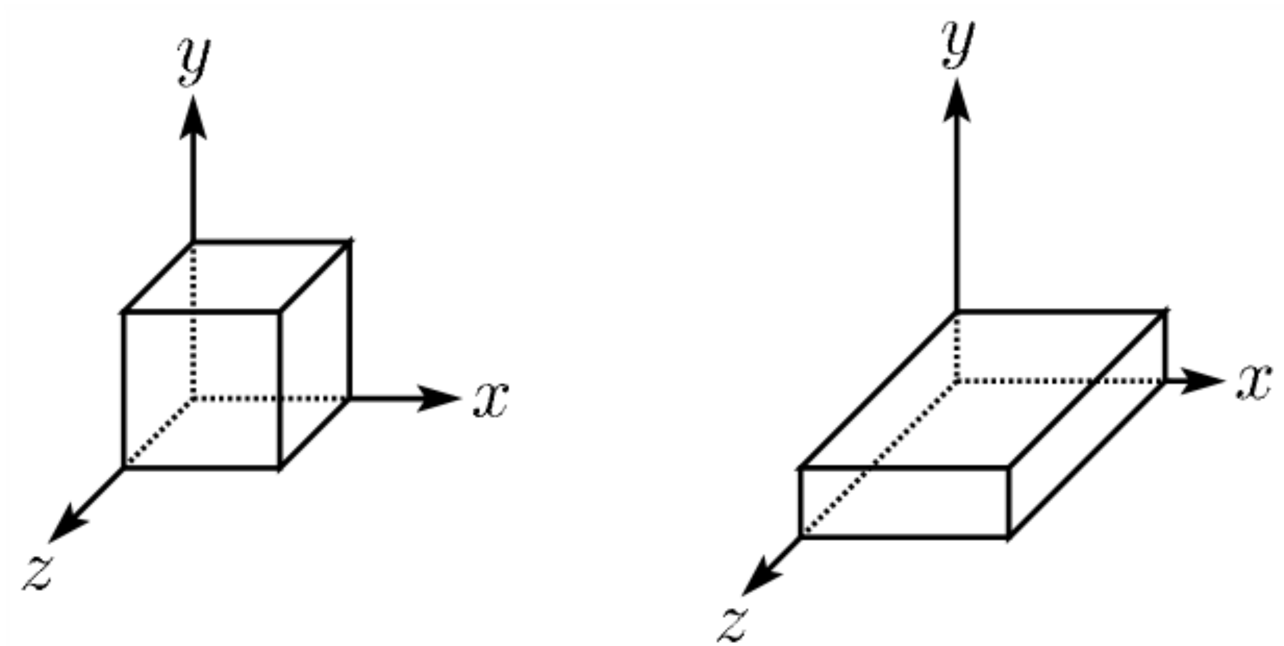


Properties of Transforms

- Compact representation
- Fast implementation
- Easy to invert
- Easy to compose

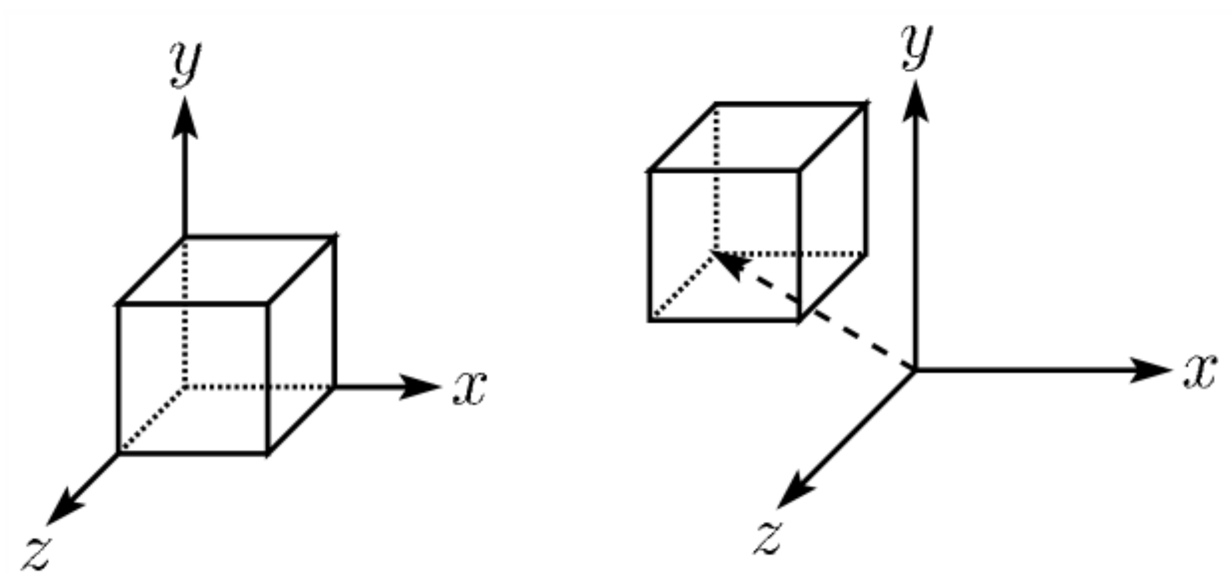
3D Scaling

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



3D Translation

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & t_x \\ 0 & 0 & 0 & t_y \\ 0 & 0 & 0 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



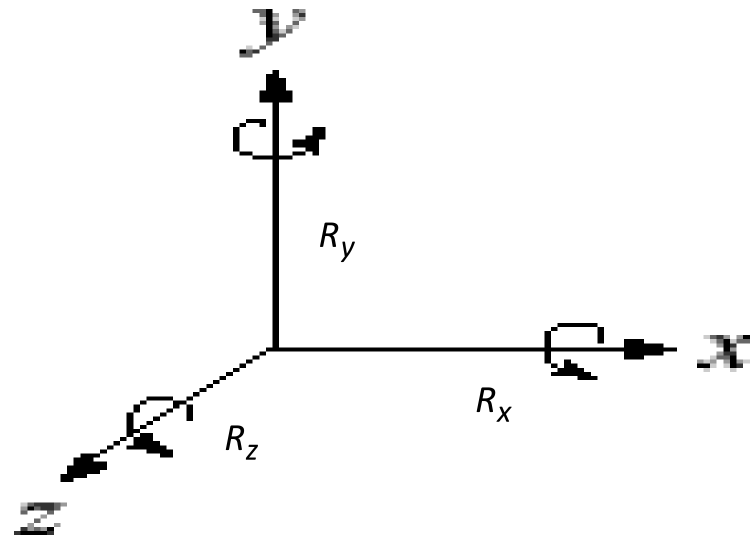
Rotation in 3D

- Rotation now has more possibilities in 3D:

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Use right hand rule

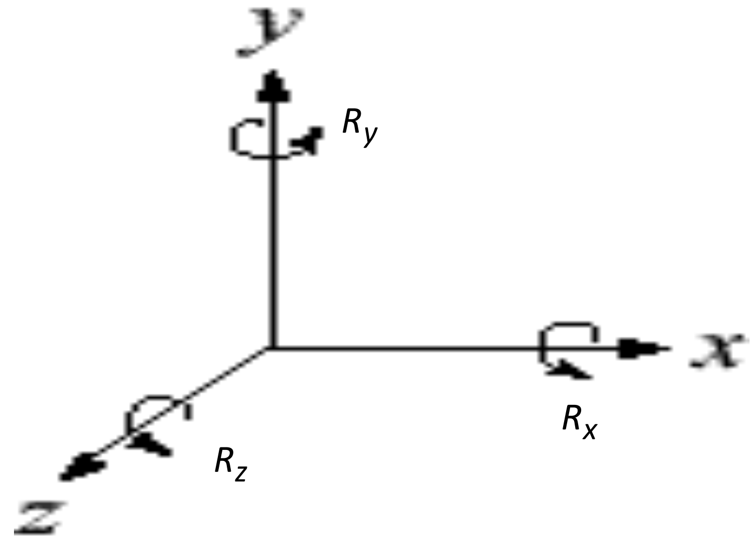
Rotation in 3D

- What about the inverses of 3D rotations?

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

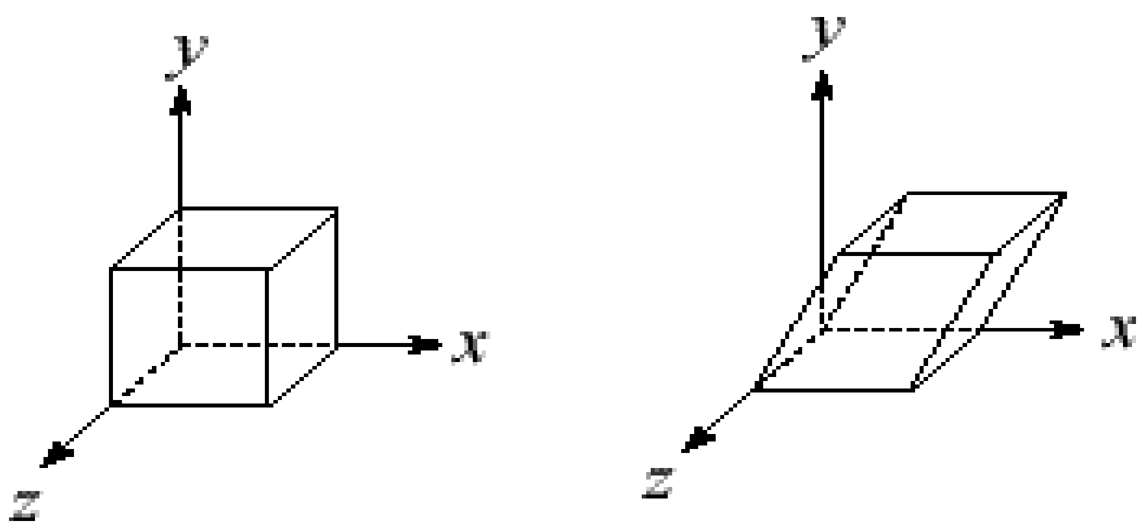
$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Shearing in 3D

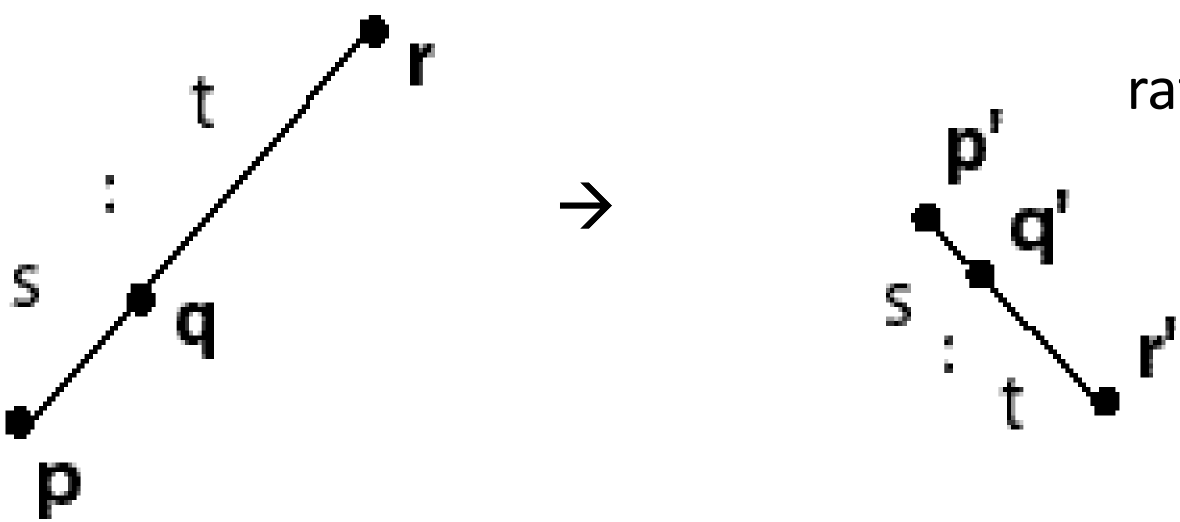
- Shearing is also more complicated. Here is one example:

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & b & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



Properties of affine transformations

- All of the transformations we've looked at so far are examples of "affine transformations."
- Here are some useful properties of affine transformations:
  - Lines map to lines
  - Parallel lines remain parallel
  - Midpoints map to midpoints (in fact, ratios are always preserved)



$$\text{ratio} = \frac{\|\mathbf{pq}\|}{\|\mathbf{qr}\|} = \frac{s}{t} = \frac{\|\mathbf{p'q'}\|}{\|\mathbf{q'r'}\|}$$

Rotation that aligns 3 orthonormal vectors with the principal axes