Affine transformations

Brian Curless
CSEP 557
Winter 2013

Reading

Required:

- Angel 3.1, 3.7-3.11

Further reading:

- Angel, the rest of Chapter 3
- Foley, et al, Chapter 5.1-5.5.
- David F. Rogers and J. Alan Adams, *Mathematical Elements for Computer Graphics*, 2nd Ed., McGraw-Hill, New York, 1990, Chapter 2.

Geometric transformations

Geometric transformations will map points in one space to points in another: $(x', y', z') = \mathbf{f}(x, y, z)$.

These transformations can be very simple, such as scaling each coordinate, or complex, such as non-linear twists and bends.

We'll focus on transformations that can be represented easily with matrix operations.

Vector representation

We can represent a **point**, **p** = (x,y), in the plane or p=(x,y,z) in 3D space

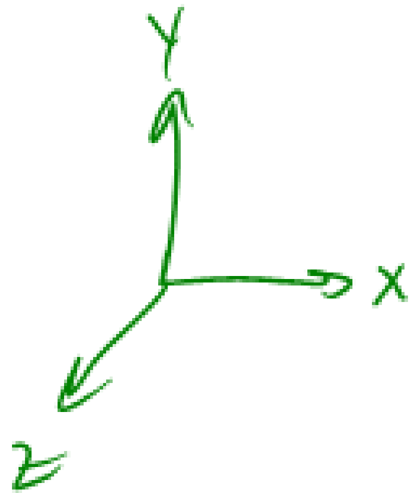
- as column vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

- as row vectors $\begin{bmatrix} x & y \end{bmatrix}$

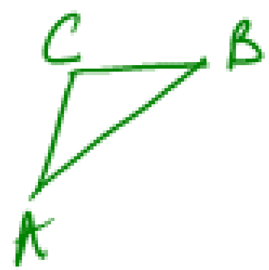
$$\begin{bmatrix} x & y & z \end{bmatrix}$$

Canonical axes

$$v = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = v_x \underset{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}{\underset{\parallel}{\hat{x}}} + v_y \underset{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}{\underset{\parallel}{\hat{y}}} + v_z \underset{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}{\underset{\parallel}{\hat{z}}}$$



right-hand rule



Vector length and dot products

$$v = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} \qquad u = \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix}$$

$$\|v\| = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

$$v \cdot u = v_x u_x + v_y u_y + v_z u_z = v^T u = \begin{bmatrix} v_x & v_y & v_z \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix}$$

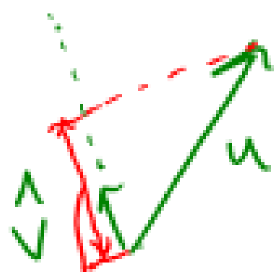
$$v \cdot u \stackrel{?}{=} u \cdot v \quad \text{True}$$

$$v \cdot v = \|v\|^2$$

$$v \cdot u = \|v\| \, \|u\| \cos\theta$$

$$v \cdot u = 0 \Rightarrow \theta = 90, -90 \quad \perp \text{ or orthogonal}$$

$$\text{or } \|u\| = 0 \text{ or } \|v\| = 0$$



$$\hat{v} = \frac{v}{\|v\|} \qquad \|\hat{v}\| = 1 \qquad \hat{v} \cdot \hat{u} = \cos\theta$$

$$\hat{v} \cdot u = \|u\| \cos\theta$$

Vector cross products

$$u \times v = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} = (u_y v_z - u_z v_y)\hat{x} + (u_z v_x - u_x v_z)\hat{y} + (u_x v_y - u_y v_x)\hat{z}$$

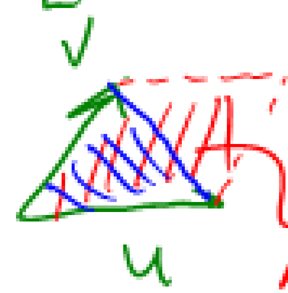
$$= \begin{bmatrix} u_y v_z - u_z v_y \\ u_z \cdots \\ u_x \cdots \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$u \times v = -v \times u$$

$$(u \times v) \cdot u = 0$$

$$\|u \times v\| = \|u\| \|v\| \sin\theta$$



Area of $\square_{uv}$ $= \|u \times v\|$

Area $\Delta_{uv} = \frac{1}{2}\|u \times v\|$

$$(AB)^T = B^T A^T$$

Representation, cont.

We can represent a **2-D transformation** M by a matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

If **p** is a column vector, M goes on the left:

$$\mathbf{p}' = M\mathbf{p}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

If **p** is a row vector, M^T goes on the right:

$$\mathbf{p}' = \mathbf{p}M^T$$

$$\begin{bmatrix} x' & y' \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} ax+by & cx+dy \end{bmatrix}$$

We will use **column vectors.**

$$(AB)^{-1}$$

$$A^{-1}A = I$$

$$(AB)^{-1}AB = I$$

$$(AB)^{-1}\cancel{A\overset{I}{BB^{-1}}}A^{-1} = \cancel{I}B^{-1}A^{-1}$$

$$(AB)^{-1} = B^{-1}A^{-1}$$

Two-dimensional transformations

Here's all you get with a 2 x 2 transformation matrix M:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

So:

$$\begin{aligned} x' &= ax + by \\ y' &= cx + dy \end{aligned}$$

We will develop some intimacy with the elements $a, b, c, d\ldots$

Identity

Suppose we choose $a=d=1$, $b=c=0$:

- Gives the **identity** matrix:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- Doesn't move the points at all

Scaling

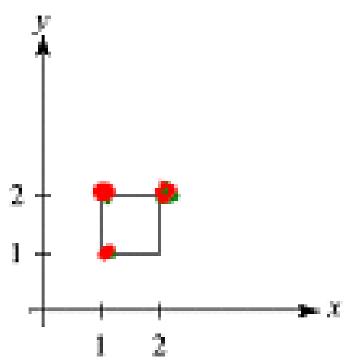
Suppose we set $b=c=0$, but let a and d take on any *positive* value:

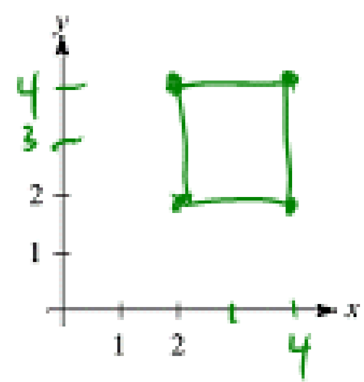
- Gives a **scaling** matrix:

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$$

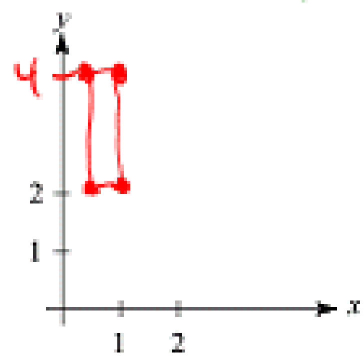
- Provides **differential (non-uniform) scaling** in x and y:

$$x' = ax$$
$$y' = dy$$





$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$



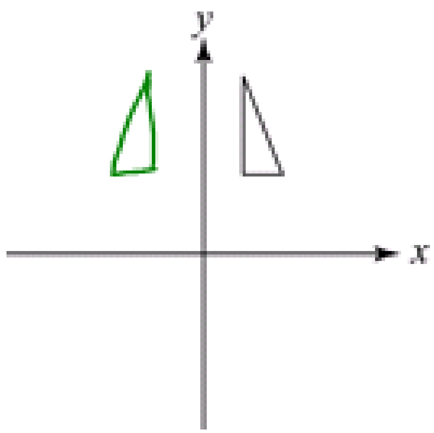
$$\begin{bmatrix} 1/2 & 0 \\ 0 & 2 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2}y \\ 2y \end{pmatrix}$$

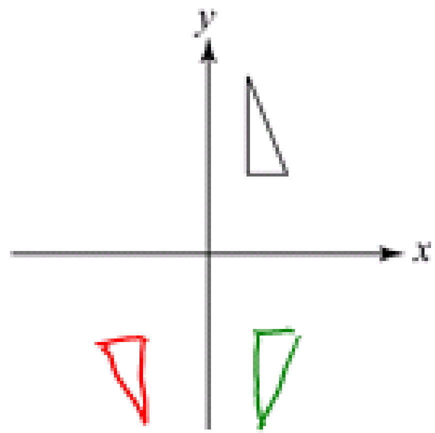
Reflection, mirror

Suppose we keep $b=c=0$, but let either a or d go negative.

Examples:

$$\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{bmatrix}-x\\ y\end{bmatrix} \qquad \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$





$$\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix} \mathrm{rot}_{180}$$

Shear

Now let's leave $a=d=1$ and experiment with b. . . .

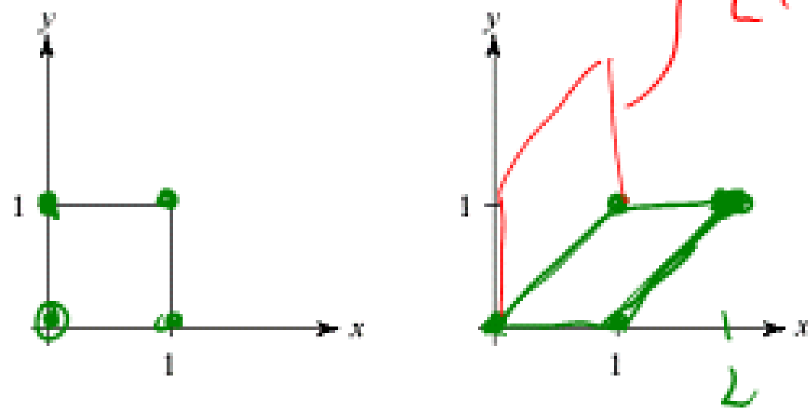
The matrix

$$\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$$

gives:

$$x' = x + by$$
$$y' = y$$

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$



$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ y \end{bmatrix}$$

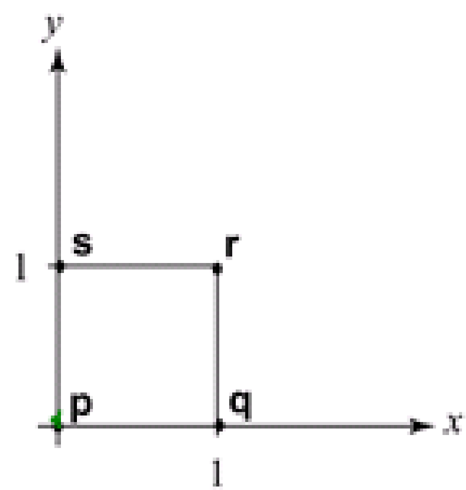
& $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

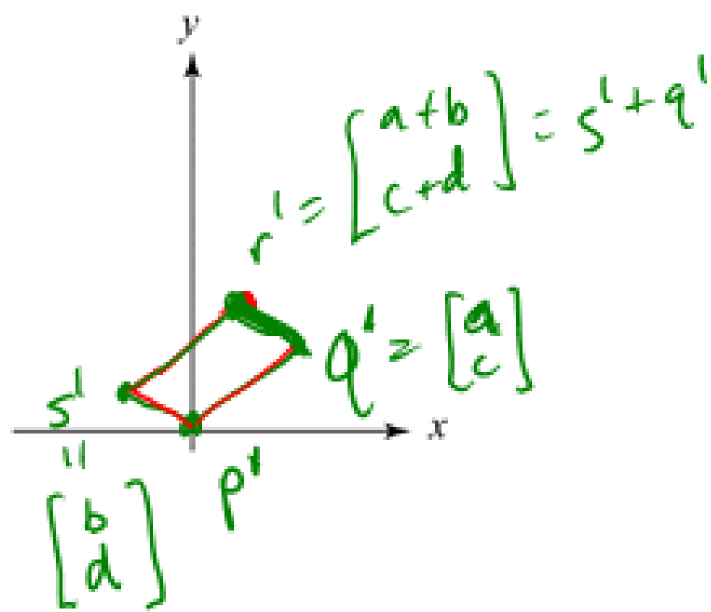
Effect on unit square

Let's see how a general 2 x 2 transformation M affects the unit square:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \mathbf{p} & \mathbf{q} & \mathbf{r} & \mathbf{s} \end{bmatrix} = \begin{bmatrix} \mathbf{p'} & \mathbf{q'} & \mathbf{r'} & \mathbf{s'} \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & a & a+b & b \\ 0 & c & c+d & d \end{bmatrix}$$





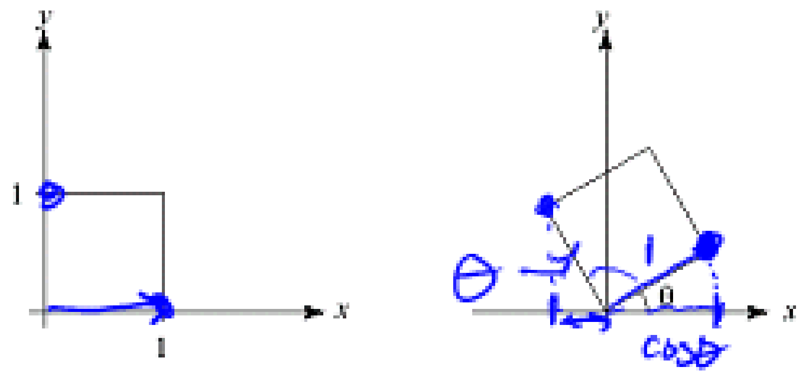
Effect on unit square, cont.

Observe:

- Origin invariant under M
- M can be determined just by knowing how the corners (1,0) and (0,1) are mapped
- a and d give x- and y-scaling
- b and c give x- and y-shearing

Rotation

From our observations of the effect on the unit square, it should be easy to write down a matrix for "rotation about the origin":



$$\cos\theta = \frac{\text{adj}}{\text{hyp}}$$

$$\cos\theta = \frac{x'}{1}$$

- $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$
- $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$

Thus,

$$M = R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Limitations of the 2 x 2 matrix

A 2 x 2 linear transformation matrix allows

- Scaling
- Rotation
- Reflection
- Shearing

Q: What important operation does that leave out?

Translation

Homogeneous coordinates

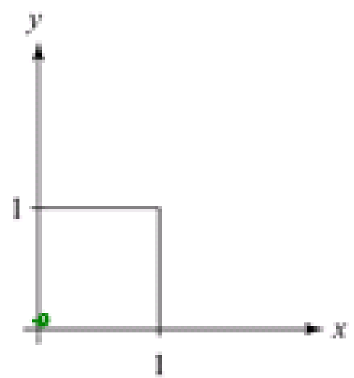
We can loft the problem up into 3-space, adding a third component to every point:

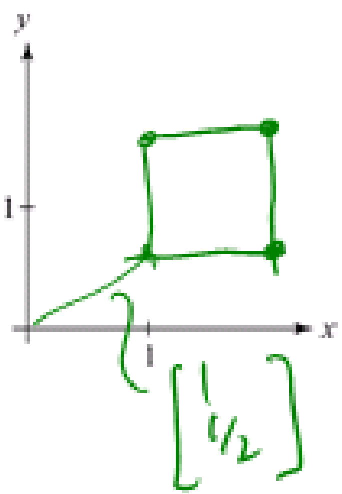
$$\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Adding the third "w" component puts us in **homogenous coordinates**.

Then, transform with a 3 x 3 matrix:

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = T(\mathbf{t}) \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix}$$





$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}$$

. . . gives **translation**!

Affine transformations

The addition of translation to linear transformations gives us **affine transformations**.

In matrix form, 2D affine transformations always look like this:

$$M = \begin{bmatrix} a & b & t_x \\ c & d & t_y \\ 0 & 0 & 1 \end{bmatrix} = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{A} & \mathbf{t} \\ \hline 0 & 0 & 1 \end{array}\right] \qquad \begin{bmatrix} p_{lin} \\ 1 \end{bmatrix} = \begin{bmatrix} Ap_{lin} + t \\ 1 \end{bmatrix}$$

2D affine transformations always have a bottom row of [0 0 1].

An "affine point" is a "linear point" with an added *w*-coordinate which is always 1:

$$\mathbf{p}_{\text{aff}} = \begin{bmatrix} \mathbf{p}_{\text{lin}} \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Applying an affine transformation gives another affine point:

$$M\mathbf{p}_{\text{aff}} = \begin{bmatrix} A\mathbf{p}_{\text{lin}} + \mathbf{t} \\ 1 \end{bmatrix}$$

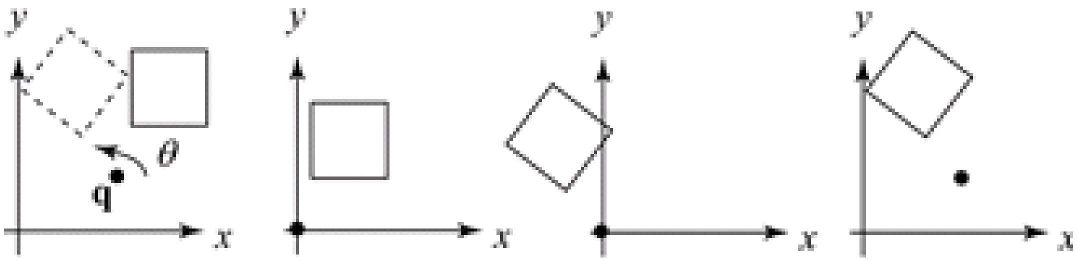
Rotation about arbitrary points

Until now, we have only considered rotation about the origin.

$T(\cdot)$

$R(\cdot)$

With homogeneous coordinates, you can specify a rotation, θ, about any point $\mathbf{q} = [q_x \; q_y \; 1]^T$ with a matrix:



$M \neq T(-q) \cdot R(\theta) \cdot T(q)$

1. Translate **q** to origin
2. Rotate
3. Translate back

$T(q)\,R(\theta)\,T(-q)\,v$

$M = T(q)\,R(\theta)\,T(-q)$

Note: Transformation order is important!!

Points and vectors

Vectors have an additional coordinate of w=0. Thus, a change of origin has no effect on vectors.

$Q - P$

$$\begin{bmatrix} Q_x \\ Q_y \\ 1 \end{bmatrix} - \begin{bmatrix} P_x \\ P_y \\ 1 \end{bmatrix} = \begin{bmatrix} \cdots \\ \cdots \\ 0 \end{bmatrix}$$

Q: What happens if we multiply a vector by an affine matrix?

$$\left[\begin{array}{c|c} A & t \\ \hline 0 & 1 \end{array}\right] \begin{bmatrix} v \\ \hline 0 \end{bmatrix} = \begin{bmatrix} Av \\ 0 \end{bmatrix}$$

These representations reflect some of the rules of affine operations on points and vectors:

vector + vector	$\rightarrow$	vector
scalar · vector	$\rightarrow$	vector
point - point	$\rightarrow$	vector
point + vector	$\rightarrow$	point
point + point	$\rightarrow$	chaos

$\alpha P + \beta Q = R$

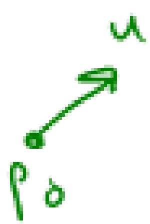
R is a point if $\alpha + \beta = 1$

R is a vec. if $\alpha + \beta = 0$

One useful combination of affine operations is:

$\alpha P + \beta Q + \gamma S = R$

$$\mathbf{p}(t) = \mathbf{p}_o + t\mathbf{u}$$



Q: What does this describe?

$t \in [-\infty, \infty] \Rightarrow p(t)$ line

$t \in [0, \infty] \Rightarrow p(t)$ ray or half-line

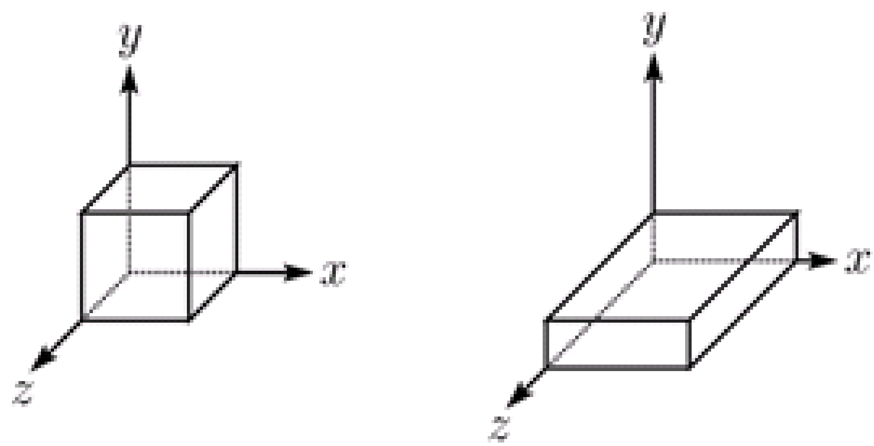
Basic 3-D transformations: scaling

Some of the 3-D affine transformations are just like the 2-D ones.

In this case, the bottom row is always [0 0 0 1].

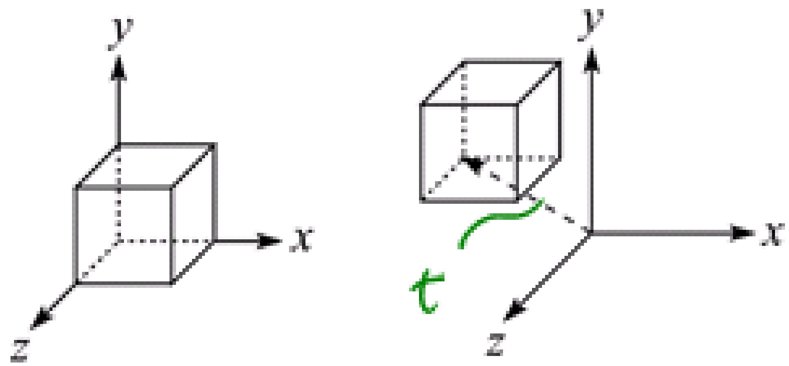
For example, scaling:

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



Translation in 3D

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



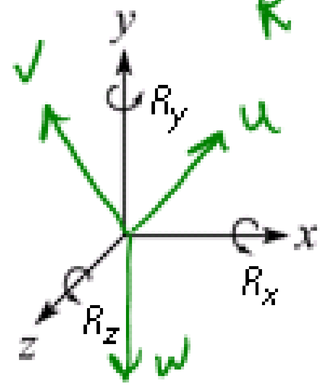
Rotation in 3D

Rotation now has more possibilities in 3D:

$$R_x(\alpha)=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & \cos\alpha & -\sin\alpha & 0\\ 0 & \sin\alpha & \cos\alpha & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$R_y(\beta)=\begin{bmatrix}\cos\beta & 0 & \sin\beta & 0\\ 0 & 1 & 0 & 0\\ -\sin\beta & 0 & \cos\beta & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$R_z(\gamma)=\begin{bmatrix}\cos\gamma & -\sin\gamma & 0 & 0\\ \sin\gamma & \cos\gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$



Use right hand rule

$$R = \begin{bmatrix} \vdots & \vdots & \vdots \\ u & v & w \\ \vdots & \vdots & \vdots \end{bmatrix}$$

$$R^T = \begin{bmatrix} \cdots & u^T & \cdots \\ \cdots & v^T & \cdots \\ \cdots & w^T & \cdots \end{bmatrix}$$

$$R^T \cdot R = \begin{bmatrix} u^Tu & u^Tv & u^Tw \\ v^Tu & v^Tv & v^Tw \\ w^Tu & w^Tv & w^Tw \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$R^T R = I$

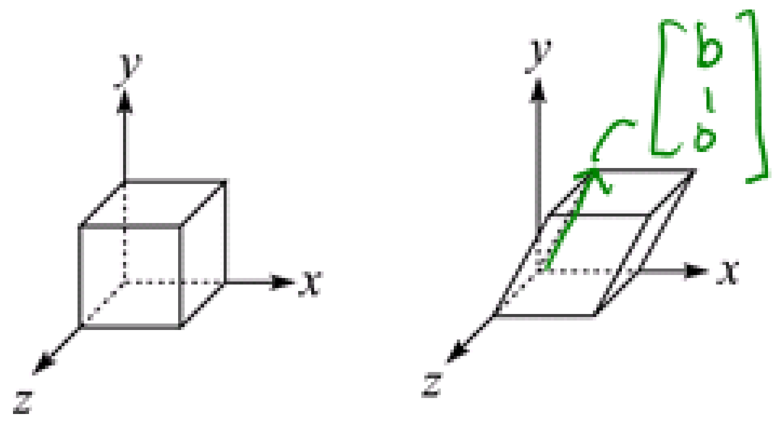
$R^T = R^{-1}$

A general rotation can be specified in terms of a prodcut of these three matrices. How else might you specify a rotation?

Shearing in 3D

Shearing is also more complicated. Here is one example:

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & b & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



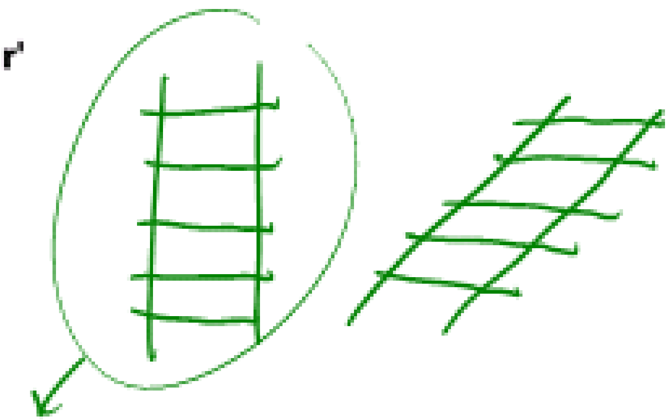
We call this a shear with respect to the x-z plane.

Properties of affine transformations

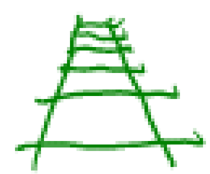
Here are some useful properties of affine transformations:

- Lines map to lines
- Parallel lines remain parallel
- Midpoints map to midpoints (in fact, ratios are always preserved)

$\ell(t) = p + tv$



$$\text{ratio} = \frac{\|\mathbf{pq}\|}{\|\mathbf{qr}\|} = \frac{s}{t} = \frac{\|\mathbf{p'q'}\|}{\|\mathbf{q'r'}\|}$$



Affine transformations in OpenGL

OpenGL maintains a "modelview" matrix that holds the current transformation **M.**

The modelview matrix is applied to points (usually vertices of polygons) before drawing.

It is modified by commands including:

- `glLoadIdentity()` $\mathbf{M} \leftarrow \mathbf{I}$
  - set **M** to identity
- `glTranslatef(`t_x`, `t_y`, `t_z`)` $\mathbf{M} \leftarrow \mathbf{MT}$
  - translate by (t_x, t_y, t_z)
- `glRotatef(θ, x, y, z)` $\mathbf{M} \leftarrow \mathbf{MR}$
  - rotate by angle θ about axis (x, y, z)
- `glScalef(`s_x`, `s_y`, `s_z`)` $\mathbf{M} \leftarrow \mathbf{MS}$
  - scale by (s_x, s_y, s_z)

Note that OpenGL adds transformations by *postmultiplication* of the modelview matrix.

Summary

What to take away from this lecture:

- All the names in boldface.
- How points and transformations are represented.
- How to compute lengths, dot products, and cross products of vectors, and what their geometrical meanings are.
- What all the elements of a 2 x 2 transformation matrix do and how these generalize to 3 x 3 transformations.
- What homogeneous coordinates are and how they work for affine transformations.
- How to concatenate transformations.
- The mathematical properties of affine transformations.