Divide and Conquer

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With Special Cameo Appearance by
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Divide and Conquer Algorithms

Split into sub problems
Recursively solve the problem
Combine solutions

Make progress in the split and combine stages
  Quicksort – progress made at the split step
  Mergesort – progress made at the combine step

D&C Algorithms
  Strassen’s Algorithm – Matrix Multiplication
  Inversions
  Median
  Closest Pair
  Integer Multiplication
  FFT
  ...
  ...
Suppose we've already invented DumbSort, taking time $n^2$

Try *Just One Level* of divide & conquer:

- DumbSort(first $n/2$ elements)
- DumbSort(last $n/2$ elements)

Merge results

Time: $2 \left(\frac{n}{2}\right)^2 + n = \frac{n^2}{2} + n \ll n^2$

*Almost twice as fast!*
Moral 1: “two halves are better than a whole”

Two problems of half size are better than one full-size problem, even given $O(n)$ overhead of recombining, since the base algorithm has super-linear complexity.

Moral 2: “If a little's good, then more's better”

Two levels of D&C would be almost 4 times faster, 3 levels almost 8, etc., even though overhead is growing. Best is usually full recursion down to some small constant size (balancing "work" vs "overhead"). In the limit: you’ve just rediscovered mergesort!
Mergesort: (recursively) sort 2 half-lists, then merge results.

\[ T(n) = 2T(n/2) + cn, \quad n \geq 2 \]

\[ T(1) = 0 \]

Solution: \( \Theta(n \log n) \)

(details later)
What you really need to know about recurrences

Work per level changes geometrically with the level

Geometrically increasing \((x > 1)\)
   The bottom level wins – count leaves

Geometrically decreasing \((x < 1)\)
   The top level wins – count top level work

Balanced \((x = 1)\)
   Equal contribution – top \(\cdot\) levels (e.g. “n log n”)
\[ T(n) = aT(n/b) + n^c \]

Balanced: \( a = b^c \)

\[ a \left( \frac{n}{b} \right)^c = a = b^c = n^c \times \log n \]

Increasing: \( a > b^c \)

Decreasing: \( a < b^c \)
Recurrences

Next: how to solve them
Mergesort: (recursively) sort 2 half-lists, then merge results.

\[ T(n) = 2T(n/2) + cn, \quad n \geq 2 \]

\[ T(1) = 0 \]

Solution: \( \Theta(n \log n) \)

(details later)

now
Solve: \[ T(1) = c \]
\[ T(n) = 2 \cdot T(n/2) + cn \]

\[ n = 2^k \quad ; \quad k = \log_2 n \]

<table>
<thead>
<tr>
<th>Level</th>
<th>Num</th>
<th>Size</th>
<th>Work</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1 = 2^0</td>
<td>n</td>
<td>cn</td>
</tr>
<tr>
<td>1</td>
<td>2 = 2^1</td>
<td>n/2</td>
<td>2cn/2</td>
</tr>
<tr>
<td>2</td>
<td>4 = 2^2</td>
<td>n/4</td>
<td>4cn/4</td>
</tr>
<tr>
<td>\ldots</td>
<td>\ldots</td>
<td>\ldots</td>
<td>\ldots</td>
</tr>
<tr>
<td>i</td>
<td>2^i</td>
<td>n/2^i</td>
<td>2^i \cdot cn/2^i</td>
</tr>
<tr>
<td>\ldots</td>
<td>\ldots</td>
<td>\ldots</td>
<td>\ldots</td>
</tr>
<tr>
<td>k-1</td>
<td>2^{k-1}</td>
<td>n/2^{k-1}</td>
<td>2^{k-1} \cdot cn/2^{k-1}</td>
</tr>
<tr>
<td>k</td>
<td>2^k</td>
<td>n/2^k = 1</td>
<td>2^k \cdot T(1)</td>
</tr>
</tbody>
</table>

Total Work: \[ c \cdot n \left(1 + \log_2 n\right) \quad \text{(add last col)} \]
Solve:

\[ T(1) = c \]

\[ T(n) = 4 \, T(n/2) + cn \]

\[ n = 2^k \quad ; \quad k = \log_2 n \]

Total Work:

\[ T(n) = \sum_{i=0}^{k} 4^i \, cn / 2^i = O(n^2) \]
Solve: \[ T(1) = c \]
\[ T(n) = 3 \cdot T(n/2) + cn \]

\[ n = 2^k ; k = \log_2 n \]

Total Work: \[ T(n) = \sum_{i=0}^{k} 3^i \frac{cn}{2^i} \]
Theorem:
\[ 1 + x + x^2 + x^3 + \ldots + x^k = \frac{(x^{k+1} - 1)}{(x - 1)} \]

proof:
\[ y = 1 + x + x^2 + x^3 + \ldots + x^k \]
\[ xy = x + x^2 + x^3 + \ldots + x^k + x^{k+1} \]
\[ xy - y = x^{k+1} - 1 \]
\[ y(x - 1) = x^{k+1} - 1 \]
\[ y = \frac{(x^{k+1} - 1)}{(x - 1)} \]
Solve: 
\[ T(1) = c \]
\[ T(n) = 3 \, T(n/2) + cn \]  (cont.)

\[
T(n) = \sum_{i=0}^{k} 3^i \frac{cn}{2^i}
\]

\[
= cn \sum_{i=0}^{k} \frac{3^i}{2^i}
\]

\[
= cn \sum_{i=0}^{k} \left(\frac{3}{2}\right)^i
\]

\[
= cn \frac{\left(\frac{3}{2}\right)^{k+1} - 1}{\left(\frac{3}{2}\right) - 1}
\]

\[
\sum_{i=0}^{k} x^i = \frac{x^{k+1} - 1}{x - 1} \quad (x \neq 1)
\]
Solve:

\[ T(1) = c \]
\[ T(n) = 3 \ T(n/2) + cn \] 

\[
\begin{align*}
\frac{cn \left( \frac{3}{2} \right)^{k+1} - 1}{\left( \frac{3}{2} \right)^{k+1} - 1} &= 2cn \left( \left( \frac{3}{2} \right)^{k+1} - 1 \right) \\
< 2cn \left( \frac{3}{2} \right)^{k+1} \\
= 3cn \left( \frac{3}{2} \right)^k \\
= 3cn \frac{3^k}{2^k}
\end{align*}
\]
Solve:

\[ T(1) = c \]

\[ T(n) = 3 \ T(n/2) + cn \quad (\text{cont.}) \]

\[
3cn \frac{3^k}{2^k} = 3cn \frac{3^\log_2 n}{2^{\log_2 n}} \\
= 3cn \frac{3^{\log_2 n}}{n} \\
= 3c3^{\log_2 n} \\
= 3c\left(n^{\log_2 3}\right) \\
= O\left(n^{1.585\ldots}\right)
\]

\[
a^{\log_b n} \\
= \left(b^{\log_b a}\right)^{\log_b n} \\
= \left(b^{\log_b n}\right)^{\log_b a} \\
= n^{\log_b a}
\]
divide and conquer – master recurrence

\[ T(n) = aT(n/b) + cn^k \text{ for } n > b \text{ then} \]

\[ a > b^k \Rightarrow T(n) = \Theta(n^{\log_b a}) \quad [\text{many subprobs} \rightarrow \text{leaves dominate}] \]

\[ a < b^k \Rightarrow T(n) = \Theta(n^k) \quad [\text{few subprobs} \rightarrow \text{top level dominates}] \]

\[ a = b^k \Rightarrow T(n) = \Theta(n^k \log n) \quad [\text{balanced} \rightarrow \text{all log n levels contribute}] \]

Fine print:
\[ a \geq 1; \; b > 1; \; c, \; d, \; k \geq 0; \; T(1) = d; \; n = b^t \text{ for some } t > 0; \]
\[ a, \; b, \; k, \; t \text{ integers. True even if it is } \left\lfloor n/b \right\rfloor \text{ instead of } n/b. \]
Expanding recurrence as in earlier examples, to get

\[ T(n) = n^h (d + cS) \]

where \( h = \log_b(a) \) (tree height) and \( S = \sum_{j=1}^{\log_b n} x^j \), where \( x = \frac{b^k}{a} \).

If \( c = 0 \) the sum \( S \) is irrelevant, and \( T(n) = O(n^h) \): all the work happens in the base cases, of which there are \( n^h \), one for each leaf in the recursion tree.

If \( c > 0 \), then the sum matters, and splits into 3 cases (like previous slide):

if \( x < 1 \), then \( S < \frac{x}{1-x} = O(1) \). [\( S \) is just the first \( \log n \) terms of the infinite series with that sum].

if \( x = 1 \), then \( S = \log_b(n) = O(\log n) \). [all terms in the sum are 1 and there are that many terms].

if \( x > 1 \), then \( S = x \cdot \frac{x^{1+\log_b(n)-1}}{x-1} \). After some algebra, \( n^h \cdot S = O(n^k) \).
Example:
Matrix Multiplication – Strassen’s Method
Multiplying Matrices

\[
\begin{bmatrix}
a_{11} & a_{12} & a_{13} & a_{14} \\
a_{21} & a_{22} & a_{23} & a_{24} \\
a_{31} & a_{32} & a_{33} & a_{34} \\
a_{41} & a_{42} & a_{43} & a_{44}
\end{bmatrix}
\cdot
\begin{bmatrix}
b_{11} & b_{12} & b_{13} & b_{14} \\
b_{21} & b_{22} & b_{23} & b_{24} \\
b_{31} & b_{32} & b_{33} & b_{34} \\
b_{41} & b_{42} & b_{43} & b_{44}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} + a_{14}b_{41} \\
a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} + a_{24}b_{41} \\
a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} + a_{34}b_{41} \\
a_{41}b_{11} + a_{42}b_{21} + a_{43}b_{31} + a_{44}b_{41}
\end{bmatrix}
\]

\[
\cdot
\begin{bmatrix}
a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} + a_{14}b_{42} \\
a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} + a_{24}b_{42} \\
a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} + a_{34}b_{42} \\
a_{41}b_{12} + a_{42}b_{22} + a_{43}b_{32} + a_{44}b_{42}
\end{bmatrix}
\]

\[
\cdot
\begin{bmatrix}
a_{11}b_{14} + a_{12}b_{24} + a_{13}b_{34} + a_{14}b_{44} \\
a_{21}b_{14} + a_{22}b_{24} + a_{23}b_{34} + a_{24}b_{44} \\
a_{31}b_{14} + a_{32}b_{24} + a_{33}b_{34} + a_{34}b_{44} \\
a_{41}b_{14} + a_{42}b_{24} + a_{43}b_{34} + a_{44}b_{44}
\end{bmatrix}
\]

\[n^3 \text{ multiplications, } n^3-n^2 \text{ additions}\]
Simple Matrix Multiply

for $i = 1$ to $n$
  for $j = 1$ to $n$
    $C[i,j] = 0$
    for $k = 1$ to $n$
      $C[i,j] = C[i,j] + A[i,k] \times B[k,j]$

$n^3$ multiplications, $n^3-n^2$ additions
Multiplying Matrices

\[
\begin{bmatrix}
    a_{11} & a_{12} & a_{13} & a_{14} \\
    a_{21} & a_{22} & a_{23} & a_{24} \\
    a_{31} & a_{32} & a_{33} & a_{34} \\
    a_{41} & a_{42} & a_{43} & a_{44}
\end{bmatrix}
\begin{bmatrix}
    b_{11} & b_{12} & b_{13} & b_{14} \\
    b_{21} & b_{22} & b_{23} & b_{24} \\
    b_{31} & b_{32} & b_{33} & b_{34} \\
    b_{41} & b_{42} & b_{43} & b_{44}
\end{bmatrix}
\]

= \[
\begin{bmatrix}
    a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} + a_{14}b_{41} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} + a_{14}b_{42} \\
    a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} + a_{24}b_{41} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} + a_{24}b_{42} \\
    a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} + a_{34}b_{41} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} + a_{34}b_{42} \\
    a_{41}b_{11} + a_{42}b_{21} + a_{43}b_{31} + a_{44}b_{41} & a_{41}b_{12} + a_{42}b_{22} + a_{43}b_{32} + a_{44}b_{42}
\end{bmatrix}
\]

\[
\begin{bmatrix}
    a_{11}b_{14} + a_{12}b_{24} + a_{13}b_{34} + a_{14}b_{44} \\
    a_{21}b_{14} + a_{22}b_{24} + a_{23}b_{34} + a_{24}b_{44} \\
    a_{31}b_{14} + a_{32}b_{24} + a_{33}b_{34} + a_{34}b_{44} \\
    a_{41}b_{14} + a_{42}b_{24} + a_{43}b_{34} + a_{44}b_{44}
\end{bmatrix}
\]
Multiplying Matrices

\[
\begin{bmatrix}
  a_{11} & a_{12} & a_{13} & a_{14} \\
  a_{21} & a_{22} & a_{23} & a_{24} \\
  a_{31} & a_{32} & a_{33} & a_{34} \\
  a_{41} & a_{42} & a_{43} & a_{44}
\end{bmatrix}
\begin{bmatrix}
  b_{11} & b_{12} & b_{13} & b_{14} \\
  b_{21} & b_{22} & b_{23} & b_{24} \\
  b_{31} & b_{32} & b_{33} & b_{34} \\
  b_{41} & b_{42} & b_{43} & b_{44}
\end{bmatrix}
= \\
\begin{bmatrix}
  a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} + a_{14}b_{41} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} + a_{14}b_{42} & \ldots & a_{11}b_{14} + a_{12}b_{24} + a_{13}b_{34} + a_{14}b_{44} \\
  a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} + a_{24}b_{41} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} + a_{24}b_{42} & \ldots & a_{21}b_{14} + a_{22}b_{24} + a_{23}b_{34} + a_{24}b_{44} \\
  a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} + a_{34}b_{41} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} + a_{34}b_{42} & \ldots & a_{31}b_{14} + a_{32}b_{24} + a_{33}b_{34} + a_{34}b_{44} \\
  a_{41}b_{11} + a_{42}b_{21} + a_{43}b_{31} + a_{44}b_{41} & a_{41}b_{12} + a_{42}b_{22} + a_{43}b_{32} + a_{44}b_{42} & \ldots & a_{41}b_{14} + a_{42}b_{24} + a_{43}b_{34} + a_{44}b_{44}
\end{bmatrix}
\]
### Multiplying Matrices

Consider two matrices $A$ and $B$:

$$
\begin{bmatrix}
  a_{11} & a_{12} & a_{13} & a_{14} \\
  a_{21} & a_{22} & a_{23} & a_{24} \\
  a_{31} & a_{32} & a_{33} & a_{34} \\
  a_{41} & a_{42} & a_{43} & a_{44}
\end{bmatrix}
\begin{bmatrix}
  b_{11} & b_{12} \\
  b_{21} & b_{22} \\
  b_{31} & b_{32} \\
  b_{41} & b_{42}
\end{bmatrix}
\end{bmatrix}.
$$

The product $A \cdot B$ is given by:

$$
\begin{bmatrix}
  a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} + a_{14}b_{41} \\
  a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} + a_{24}b_{41} \\
  a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} + a_{34}b_{41} \\
  a_{41}b_{11} + a_{42}b_{21} + a_{43}b_{31} + a_{44}b_{41}
\end{bmatrix}
\begin{bmatrix}
  a_{11} & a_{12} & a_{13} & a_{14} \\
  a_{21} & a_{22} & a_{23} & a_{24} \\
  a_{31} & a_{32} & a_{33} & a_{34} \\
  a_{41} & a_{42} & a_{43} & a_{44}
\end{bmatrix}
\begin{bmatrix}
  b_{11} & b_{12} \\
  b_{21} & b_{22} \\
  b_{31} & b_{32} \\
  b_{41} & b_{42}
\end{bmatrix}.
$$

$$
\begin{bmatrix}
  a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} + a_{14}b_{41} \\
  a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} + a_{24}b_{41} \\
  a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} + a_{34}b_{41} \\
  a_{41}b_{11} + a_{42}b_{21} + a_{43}b_{31} + a_{44}b_{41}
\end{bmatrix}
\begin{bmatrix}
  a_{11} & a_{12} & a_{13} & a_{14} \\
  a_{21} & a_{22} & a_{23} & a_{24} \\
  a_{31} & a_{32} & a_{33} & a_{34} \\
  a_{41} & a_{42} & a_{43} & a_{44}
\end{bmatrix}
\begin{bmatrix}
  b_{11} & b_{12} \\
  b_{21} & b_{22} \\
  b_{31} & b_{32} \\
  b_{41} & b_{42}
\end{bmatrix}.
$$
Counting arithmetic operations:

\[ T(n) = 8T(n/2) + 4(n/2)^2 = 8T(n/2) + n^2 \]
\[ T(n) = \begin{cases} 
1 & \text{if } n = 1 \\
8T(n/2) + n^2 & \text{if } n > 1 
\end{cases} \]

By Master Recurrence, if
\[ T(n) = aT(n/b) + cn^k \text{ and } a > b^k \]
then
\[ T(n) = \Theta(n^{\log_b a}) = \Theta(n^{\log_2 8}) = \Theta(n^3) \]
Strassen’s algorithm

Strassen’s algorithm

Multiply 2x2 matrices using 7 instead of 8 multiplications (and lots more than 4 additions)

\[ T(n) = 7 \cdot T(n/2) + cn^2 \]

7 > 2^2 so \( T(n) \) is \( \Theta(\log_27) \) which is \( O(n^{2.81}) \)

Asymptotically fastest known algorithm uses \( O(n^{2.376}) \) time

not practical but Strassen’s may be practical provided calculations are exact and we stop recursion when matrix has size about 100 (maybe 10)
The algorithm

\[ P_1 = A_{12}(B_{11} + B_{21}) \]
\[ P_2 = A_{21}(B_{12} + B_{22}) \]
\[ P_3 = (A_{11} - A_{12})B_{11} \]
\[ P_4 = (A_{22} - A_{21})B_{22} \]
\[ P_5 = (A_{22} - A_{12})(B_{21} - B_{22}) \]
\[ P_6 = (A_{11} - A_{21})(B_{12} - B_{11}) \]
\[ P_7 = (A_{21} - A_{12})(B_{11} + B_{22}) \]

\[ C_{11} = P_1 + P_3 + P_7 \]
\[ C_{12} = P_2 + P_3 + P_6 - P_7 \]
\[ C_{21} = P_1 + P_4 + P_5 + P_7 \]
\[ C_{22} = P_2 + P_4 \]
Example: Counting Inversions
Inversion Problem

Let \( a_1, \ldots, a_n \) be a permutation of \( 1 \ldots n \).

\((a_i, a_j)\) is an inversion if \( i < j \) and \( a_i > a_j \).

\[4, 6, 1, 7, 3, 2, 5\]

Problem: given a permutation, count the number of inversions.

This can be done easily in \( O(n^2) \) time.

Can we do better?
Counting inversions can be used to measure closeness of ranked preferences.

People rank 20 movies, based on their rankings you cluster people who like the same types of movies.

Can also be used to measure nonlinear correlation.
Let $a_1, \ldots, a_n$ be a permutation of $1 \ldots n$

$(a_i, a_j)$ is an inversion if $i < j$ and $a_i > a_j$

4, 6, 1, 7, 3, 2, 5

Problem: given a permutation, count the number of inversions

This can be done easily in $O(n^2)$ time

Can we do better?
Counting Inversions

Count inversions on lower half
Count inversions on upper half
Count the inversions between the halves
Problem – how do we count inversions between sub problems in $O(n)$ time?

Solution – Count inversions while merging

Standard merge algorithm – add to inversion count when an element is moved from the upper array to the solution.
Counting inversions while merging

Indicate the number of inversions for each element detected when merging
Counting inversions between two sorted lists
O(1) per element to count inversions

Algorithm summary
Satisfies the "Standard recurrence"
T(n) = 2 T(n/2) + cn
A Divide & Conquer Example: Closest Pair of Points
Given n points and *arbitrary* distances between them, find the closest pair. (E.g., think of distance as airfare – definitely *not* Euclidean distance!)

Must look at all n choose 2 pairwise distances, else any one you didn’t check might be the shortest.

Also true for Euclidean distance in 1-2 dimensions?
closest pair of points: 1 dimensional version

Given n points on the real line, find the closest pair

Ç

Closest pair is *adjacent* in ordered list
Time $O(n \log n)$ to sort, if needed
Plus $O(n)$ to scan adjacent pairs
Key point: do *not* need to calc distances between all pairs: exploit geometry + ordering
closest pair of points: 2 dimensional version

Closest pair. Given n points in the plane, find a pair with smallest Euclidean distance between them.

Fundamental geometric primitive.
  Graphics, computer vision, geographic information systems, molecular modeling, air traffic control.
  Special case of nearest neighbor, Euclidean MST, Voronoi.
  fast closest pair inspired fast algorithms for these problems

Brute force. Check all pairs of points $p$ and $q$ with $\Theta(n^2)$ comparisons.

1-D version. $O(n \log n)$ easy if points are on a line.

Assumption. No two points have same $x$ coordinate. (Just to simplify presentation)
Closest pair of points. 2d, Euclidean distance: 1st try

Divide. Sub-divide region into 4 quadrants.
Divide. Sub-divide region into 4 quadrants.
Obstacle. Impossible to ensure n/4 points in each piece.
Algorithm.

Divide: draw vertical line $L$ with $\approx n/2$ points on each side.
Algorithm.

Divide: draw vertical line L with ≈ n/2 points on each side.

Conquer: find closest pair on each side, recursively.
Algorithm.
Divide: draw vertical line $L$ with $\approx n/2$ points on each side.
Conquer: find closest pair on each side, recursively.
Combine: find closest pair with one point in each side.
Return best of 3 solutions.
Find closest pair with one point in each side, *assuming* distance $< \delta$. 

\[ \delta = \min(12, 21) \]
Find closest pair with one point in each side, *assuming distance* $< \delta$.

Observation: suffices to consider points within $\delta$ of line $L$. 

\[ \delta = \min(12, 21) \]
Find closest pair with one point in each side, *assuming* distance $< \delta$.

Observation: suffices to consider points within $\delta$ of line L. Almost the one-D problem again: Sort points in $2\delta$-strip by their y coordinate.
Find closest pair with one point in each side, \textit{assuming} distance \(< \delta\).

Observation: suffices to consider points within \(\delta\) of line \(L\). Almost the one-D problem again: Sort points in \(2\delta\)-strip by their \(y\) coordinate. Only check pts within 8 in sorted list!

\[ \delta = \min(12, 21) \]
Def. Let $s_i$ have the $i^{th}$ smallest y-coordinate among points in the $2\delta$-width-strip.

Claim. If $|i - j| > 8$, then the distance between $s_i$ and $s_j$ is $> \delta$.

Pf: No two points lie in the same $\frac{1}{2}\delta$-by-$\frac{1}{2}\delta$ box:

$$\sqrt{\frac{1}{2}^2 + \frac{1}{2}^2} = \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2} \approx 0.7 < 1$$

so $\leq 8$ boxes within $+\delta$ of $y(s_i)$. 
Closest-Pair(p₁, ..., pₙ) {
    if(n <= ??) return ??
    Compute separation line L such that half the points are on one side and half on the other side.

    δ₁ = Closest-Pair(left half)
    δ₂ = Closest-Pair(right half)
    δ = min(δ₁, δ₂)

    Delete all points further than δ from separation line L

    Sort remaining points p[1]..p[m] by y-coordinate.

    for i = 1..m
        k = 1
        while i+k <= m && p[i+k].y < p[i].y + δ
            δ = min(δ, distance between p[i] and p[i+k]);
            k++;

    return δ.
}
Analysis, I: Let $D(n)$ be the number of pairwise distance calculations in the Closest-Pair Algorithm when run on $n \geq 1$ points.

$$D(n) \leq \begin{cases} 0 & n = 1 \\ 2D(n/2) + 7n & n > 1 \end{cases} \Rightarrow D(n) = O(n \log n)$$

BUT – that’s only the number of distance calculations.

What if we counted comparisons?
closest pair of points: analysis

Analysis, II: Let $C(n)$ be the number of comparisons between coordinates/distances in the Closest-Pair Algorithm when run on $n \geq 1$ points

$$
C(n) \leq \begin{cases} 
0 & n=1 \\
2C(n/2) + kn \log n & n > 1 
\end{cases} 
\Rightarrow C(n) = O(n \log^2 n)
$$

for some constant $k$

Q. Can we achieve $O(n \log n)$?

A. Yes. Don't sort points from scratch each time.
Sort by $x$ at top level only.
Each recursive call returns $\delta$ and list of all points sorted by $y$
Sort by merging two pre-sorted lists.

$$
T(n) \leq 2T(n/2) + O(n) \Rightarrow T(n) = O(n \log n)
$$
Code is longer & more complex
$O(n \log n)$ vs $O(n^2)$ may hide 10x in constant?
How many points?

<table>
<thead>
<tr>
<th>$n$</th>
<th>Speedup: $n^2 / (10 \cdot n \log_2 n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.3</td>
</tr>
<tr>
<td>100</td>
<td>1.5</td>
</tr>
<tr>
<td>1,000</td>
<td>10</td>
</tr>
<tr>
<td>10,000</td>
<td>75</td>
</tr>
<tr>
<td>100,000</td>
<td>602</td>
</tr>
<tr>
<td>1,000,000</td>
<td>5,017</td>
</tr>
<tr>
<td>10,000,000</td>
<td>43,004</td>
</tr>
</tbody>
</table>
Going From Code to Recurrence
Carefully define what you’re counting, and write it down!

“Let $C(n)$ be the number of comparisons between sort keys used by MergeSort when sorting a list of length $n \geq 1$”

In code, clearly separate base case from recursive case, highlight recursive calls, and operations being counted.

Write Recurrence(s)
merge sort

Base Case

MS(A: array[1..n]) returns array[1..n] {
    If(n=1) return A;
    New L:array[1:n/2] = MS(A[1..n/2]);
    New R:array[1:n/2] = MS(A[n/2+1..n]);
    Return(Merge(L,R));
}

Merge(A,B: array[1..n]) {
    New C: array[1..2n];
    a=1; b=1;
    For i = 1 to 2n {
        C[i] = “smaller of A[a], B[b] and a++ or b++”;
    }
    Return C;
}
Total time: proportional to $C(n)$
(loops, copying data, parameter passing, etc.)
Carefully define what you’re counting, and write it down!

“Let $D(n)$ be the number of pairwise distance calculations in the Closest-Pair Algorithm when run on $n \geq 1$ points.”

In code, clearly separate base case from recursive case, highlight recursive calls, and operations being counted.

Write Recurrence(s)
Closest Pair \( (p_1, \ldots, p_n) \) {
    if \( n \leq 1 \) return \( \infty \)

    Compute separation line \( L \) such that half the points are on one side and half on the other side.

    \[ \delta_1 = \text{Closest-Pair(left half)} \]
    \[ \delta_2 = \text{Closest-Pair(right half)} \]
    \[ \delta = \min(\delta_1, \delta_2) \]

    Delete all points further than \( \delta \) from separation line \( L \).

    Sort remaining points \( p[1], \ldots, p[m] \) by y-coordinate.

    for \( i = 1 \ldots m \)
        \( k = 1 \)
        while \( i+k \leq m \) and \( p[i+k].y < p[i].y + \delta \)
            \[ \delta = \min(\delta, \text{distance between } p[i] \text{ and } p[i+k]) \]
            \( k++ \)
    return \( \delta \).
Analysis, I: Let $D(n)$ be the number of pairwise distance calculations in the Closest-Pair Algorithm when run on $n \geq 1$ points

$$D(n) \leq \begin{cases} 0 & n = 1 \\ 2D(n/2) + 7n & n > 1 \end{cases} \implies D(n) = O(n \log n)$$

BUT – that’s only the number of distance calculations

What if we counted comparisons?
Carefully define what you’re counting, and write it down!

“Let $D(n)$ be the number of comparisons between coordinates/distances in the Closest-Pair Algorithm when run on $n \geq 1$ points”

In code, clearly separate base case from recursive case, highlight recursive calls, and operations being counted.

Write Recurrence(s)
The closest pair algorithm

```
Closest-Pair(p₁, ..., pₙ) {
    if (n ≤ 1) return ∞

    Compute separation line L such that half the points are on one side and half on the other side.

    δ₁ = Closest-Pair(left half)
    δ₂ = Closest-Pair(right half)
    δ = min(δ₁, δ₂)

    Delete all points further than δ from separation line L.

    Sort remaining points p[1..m] by y-coordinate.

    for i = 1..m
        k = 1
        while i+k ≤ m && p[i+k].y < p[i].y + δ
            δ = min(δ, distance between p[i] and p[i+k]);
            k++;
    return δ.
}
```
Analysis, II: Let $C(n)$ be the number of comparisons of coordinates/distances in the Closest-Pair Algorithm when run on $n \geq 1$ points.

$$
C(n) \leq \begin{cases} 
0 & n = 1 \\
2C(n/2) + k_4 n \log n & n > 1 
\end{cases}
\Rightarrow C(n) = O(n \log^2 n)
$$

for some $k_4 \leq k_1 + k_2 + k_3 + 7$

Q. Can we achieve time $O(n \log n)$?

A. Yes. Don't sort points from scratch each time.
   Sort by $x$ at top level only.
   Each recursive call returns a list of all points sorted by $y$.
   Sort by merging two pre-sorted lists.

$$
T(n) \leq 2T(n/2) + O(n) \Rightarrow T(n) = O(n \log n)
$$
Integer Multiplication
Add. Given two n-bit integers \( a \) and \( b \), compute \( a + b \).

\[ \begin{array}{c}
1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
+ & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\
\hline
1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
\end{array} \]

\( O(n) \) bit operations.
Add. Given two $n$-bit integers $a$ and $b$, compute $a + b$. \(\mathcal{O}(n)\) bit operations.

Multiply. Given two $n$-bit integers $a$ and $b$, compute $a \times b$. The “grade school” method: \(\Theta(n^2)\) bit operations.
To multiply two 2-digit integers:
Multiply four 1-digit integers.
Add, shift some 2-digit integers to obtain result.

\[
x = 10 \cdot x_1 + x_0 \\
y = 10 \cdot y_1 + y_0 \\
xy = (10 \cdot x_1 + x_0) (10 \cdot y_1 + y_0) \\
= 100 \cdot x_1y_1 + 10 \cdot (x_1y_0 + x_0y_1) + x_0y_0
\]

Same idea works for long integers – can split them into 4 half-sized ints
To multiply two n-bit integers:

Multiply four \( \frac{1}{2}n \)-bit integers.

Add two \( \frac{1}{2}n \)-bit integers, and shift to obtain result.

\[
\begin{align*}
x &= 2^{n/2} \cdot x_1 + x_0 \\
y &= 2^{n/2} \cdot y_1 + y_0 \\
xy &= \left(2^{n/2} \cdot x_1 + x_0\right) \left(2^{n/2} \cdot y_1 + y_0\right) \\
 &= 2^n \cdot x_1 y_1 + 2^{n/2} \left(x_1 y_0 + x_0 y_1\right) + x_0 y_0
\end{align*}
\]

\[
T(n) = 4T(n/2) + \Theta(n) \Rightarrow T(n) = \Theta(n^2)
\]

assumes \( n \) is a power of 2
key trick: 2 multiplies for the price of 1:

\[
\begin{align*}
x &= 2^{n/2} \cdot x_1 + x_0 \\
y &= 2^{n/2} \cdot y_1 + y_0 \\
xy &= (2^{n/2} \cdot x_1 + x_0)(2^{n/2} \cdot y_1 + y_0) \\
&= 2^n \cdot x_1 y_1 + 2^{n/2} \cdot (x_1 y_0 + x_0 y_1) + x_0 y_0
\end{align*}
\]

Well, ok, 4 for 3 is more accurate...

\[
\begin{align*}
\alpha &= x_1 + x_0 \\
\beta &= y_1 + y_0 \\
\alpha \beta &= (x_1 + x_0)(y_1 + y_0) \\
&= x_1 y_1 + (x_1 y_0 + x_0 y_1) + x_0 y_0 \\
(x_1 y_0 + x_0 y_1) &= \alpha \beta - x_1 y_1 - x_0 y_0
\end{align*}
\]
Karatsuba multiplication

To multiply two n-bit integers:
Add two \( \frac{1}{2}n \) bit integers.
Multiply three \( \frac{1}{2}n \)-bit integers.
Add, subtract, and shift \( \frac{1}{2}n \)-bit integers to obtain result.

\[
\begin{align*}
  x &= 2^{n/2} \cdot x_1 + x_0 \\
  y &= 2^{n/2} \cdot y_1 + y_0 \\
  xy &= 2^n \cdot x_1 y_1 + 2^{n/2} \cdot (x_1 y_0 + x_0 y_1) + x_0 y_0 \\
       &= 2^n \cdot x_1 y_1 + 2^{n/2} \cdot ((x_1 + x_0)(y_1 + y_0) - x_1 y_1 - x_0 y_0) + x_0 y_0
\end{align*}
\]

Theorem. \[\text{[Karatsuba-Ofman, 1962]} \] Can multiply two n-digit integers in \(O(n^{1.585})\) bit operations.

\[
T(n) \leq T(\lceil n/2 \rceil) + T(\lceil n/2 \rceil) + T(1 + \lceil n/2 \rceil) + \Theta(n)
\]

Sloppy version: \(T(n) \leq 3T(n/2) + O(n)\)

\[\Rightarrow T(n) = O(n^{\log_2 3}) = O(n^{1.585})\]
Theorem. [Karatsuba-Ofman, 1962] Can multiply two n-digit integers in $O(n^{1.585})$ bit operations.

$$T(n) \leq T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + T(1+\lfloor n/2 \rfloor) + \Theta(n)$$

(recursive calls)

(add, subtract, shift)

Sloppy version: $T(n) \leq 3T(n/2) + O(n)$

$\Rightarrow T(n) = O(n \log_3 2) = O(n^{1.585})$

$n \rightarrow \text{next bigger power of 2}$

$\lfloor \frac{x}{2} \rfloor + 1 \rightarrow \frac{x}{2}$

$T(n) \leq 3T(\lfloor n/2 \rfloor) + O(n)$
Naïve: $\Theta(n^2)$
Karatsuba: $\Theta(n^{1.59...})$

Amusing exercise: generalize Karatsuba to do 5 size
$n/3$ subproblems $\rightarrow \Theta(n^{1.46...})$

Best known: $\Theta(n \log n \log\log n)$
"Fast Fourier Transform"
but mostly unused in practice (unless you need really big numbers - a billion digits of $\pi$, say)

High precision arithmetic is important for crypto
Polynomial Multiplication
Another D&C Example: Multiplying Polynomials

Similar ideas apply to polynomial multiplication

We’ll describe the basic ideas by multiplying polynomials rather than integers
In fact, it’s somewhat simpler: no carries!
These are just formal sequences of coefficients so when we show something multiplied by $x^k$ it just means shifted $k$ places to the left – basically no work.

Usual Polynomial Multiplication:

\[
\begin{array}{c}
3x^2 + 2x + 2 \\
\times (x^2 - 3x + 1) \\
\hline
3x^2 + 2x + 2 \\
\hline
-9x^3 - 6x^2 - 6x \\
3x^4 + 2x^3 + 2x^2 \\
\hline
3x^4 - 7x^3 - x^2 - 4x + 2
\end{array}
\]
Polynomial Multiplication

Given:
Degree m-1 polynomials P and Q

\[ P = a_0 + a_1 x + a_2 x^2 + \ldots + a_{m-2} x^{m-2} + a_{m-1} x^{m-1} \]
\[ Q = b_0 + b_1 x + b_2 x^2 + \ldots + b_{m-2} x^{m-2} + b_{m-1} x^{m-1} \]

Compute:
Degree 2m-2 Polynomial P Q

\[ P Q = a_0 b_0 + (a_0 b_1 + a_1 b_0) x + (a_0 b_2 + a_1 b_1 + a_2 b_0) x^2 + \ldots + (a_{m-2} b_{m-1} + a_{m-1} b_{m-2}) x^{2m-3} + a_{m-1} b_{m-1} x^{2m-2} \]

Obvious Algorithm:
Compute all \( a_i b_j \) and collect terms
\( \Theta (m^2) \) time
Assume \( m=2k \)

\[
\begin{align*}
P &= (a_0 + a_1 x + a_2 x^2 + \ldots + a_{k-2} x^{k-2} + a_{k-1} x^{k-1}) + \\
   &= P_0 + P_1 x^k \\
Q &= Q_0 + Q_1 x^k
\end{align*}
\]

\[
P \cdot Q = (P_0 + P_1 x^k)(Q_0 + Q_1 x^k) = P_0 Q_0 + (P_1 Q_0 + P_0 Q_1) x^k + P_1 Q_1 x^{2k}
\]

4 sub-problems of size \( k=m/2 \) plus linear combining

\[
T(m) = 4T(m/2) + cm
\]

Solution \( T(m) = O(m^2) \)
Karatsuba’s Algorithm

A better way to compute terms

Compute

\[ P_0 Q_0 \]
\[ P_1 Q_1 \]
\[ (P_0 + P_1)(Q_0 + Q_1) \] which is \[ P_0 Q_0 + P_1 Q_0 + P_0 Q_1 + P_1 Q_1 \]

Then

\[ P_0 Q_1 + P_1 Q_0 = (P_0 + P_1)(Q_0 + Q_1) - P_0 Q_0 - P_1 Q_1 \]

3 sub-problems of size \( m/2 \) plus \( O(m) \) work

\[ T(m) = 3 T(m/2) + cm \]

\[ T(m) = O(m^\alpha) \] where \( \alpha = \log_2 3 = 1.585... \)
PolyMul(P, Q):

// P, Q are length m = 2k vectors, with P[i], Q[i] being
// the coefficient of x^i in polynomials P, Q respectively.
if (m == 1) return (P[0]*Q[0]);

Let Pzero be elements 0..k-1 of P; Pone be elements k..m-1
Qzero, Qone : similar

Prod1 = PolyMul(Pzero, Qzero);       // result is a (2k-1)-vector
Prod2 = PolyMul(Pone, Qone);         // ditto
Pzo = Pzero + Pone;                  // add corresponding
         elements
Qzo = Qzero + Qone;                  // ditto
Prod3 = PolyMul(Pzo, Qzo);           // another (2k-1)-vector
Mid = Prod3 - Prod1 - Prod2;         // subtract corr. elements
R = Prod1 + Shift(Mid, m/2) + Shift(Prod2, m) // a (2m-1)-vector
Return( R );
Multiplication – The Bottom Line

Polynomials

Naïve: $\Theta(n^2)$
Karatsuba: $\Theta(n^{1.585...})$
Best known: $\Theta(n \log n)$
"Fast Fourier Transform"

Integers

Similar, but some ugly details re: carries, etc.
gives $\Theta(n \log n \log \log \log n)$,
but mostly unused in practice
Median and Selection
Computing the Median

Median: Given $n$ numbers, find the number of rank $n/2$ (to be precise, say: $\lfloor n/2 \rfloor$)

Selection: given $n$ numbers and an integer $k$, find the $k$-th largest

E.g., Median is $\lfloor n/2 \rfloor$-nd largest
Can find max with n-1 comparisons
Can find 2\textsuperscript{nd} largest with another n-2
3\textsuperscript{rd} largest with another n-3
etc.: k\textsuperscript{th} largest in \(O(kn)\)

What about \(k > \log n\)?

Can we do better?
Select(A, k) {
    Choose x from A
    \[ S_1 = \{ y \in A \mid y < x \} \]
    \[ S_2 = \{ y \in A \mid y = x \} \]
    \[ S_3 = \{ y \in A \mid y > x \} \]
    if (|S_1| \geq k)
        return Select(S_1, k)
    else if (|S_1| + |S_2| \geq k)
        return x
    else
        return Select(S_3, k - |S_1| - |S_2|)
}
Randomized Selection

Choose the element at random

Analysis (not here) can show that the algorithm has expected run time $O(n)$

Sketch: a random element eliminates, on average, $\sim \frac{1}{2}$ of the data

Although worst case is $\Theta(n^2)$, albeit improbable (like Quicksort), for most purposes this is the method of choice

Worst case matters? Read on...
What is the run time of select if we can guarantee that “choose” finds an x such that $|S_1| < 3n/4$ and $|S_3| < 3n/4$
A very clever “choose” algorithm . . .

Split into $n/5$ sets of size 5
$M$ be the set of medians of these sets
Return $x = \text{the median of } M$
Split into \( \frac{n}{5} \) sets of size 5
Let \( M \) be the set of medians of these sets
Choose \( x \) to be the median of \( M \)
Construct \( S_1, S_2 \) and \( S_3 \) as above
Recursive call in \( S_1 \) or \( S_3 \)

To show: \( |S_1| < \frac{3n}{4}, \ |S_3| < \frac{3n}{4} \)

\( \frac{n}{5} + \frac{3n}{4} = 0.95n \Rightarrow O(n), \) worst case
x = median of medians

NB: conceptual; algorithm finds median(s), but does not sort
Median of Medians

Points $\geq x$, \[ \therefore \text{NOT in } S_1 \]
\[ \approx 3n/10 \text{ of them} \]

Points $\leq x$, \[ \therefore \text{NOT in } S_3 \]
\[ \approx 3n/10 \text{ of them} \]

$x = \text{median of medians}$

Bottom Line: recursive call on $S_1$ or $S_3$ includes only about 70% of points

NB: conceptual; algorithm finds median(s), but does not sort
\approx \frac{7n}{10} \text{ points in subproblem}

More precisely, various fussiness:

\left\lfloor \frac{n}{5} \right\rfloor \text{ groups, all but (possibly) last of size 5}

Upper/lower half of \geq \left\lfloor \frac{n}{5} \right\rfloor / 2 \text{ groups excluded}

With some algebra, \exists a, b, c \text{ such that:}

T(n) \leq T(\frac{7n}{10} + a) + T(\frac{n}{5} + b) + c \cdot n
BFPRT Recurrence

\[ T(n) \leq T(7n/10 + a) + T(n/5 + b) + cn \]

\[ \leq 20c \left( \frac{7n}{10} + a \right) + 20c \left( \frac{n}{5} + b \right) + cn \]

\[ = 14cn + 4cn + cn + 20c(a + b) \]

\[ = 19cn + 20c(\frac{a + b}{n}) \cdot cn \]

\[ \leq 20cn \]

Base case: \( n \leq 20c(a+b) \),

\[ c = \text{Max}(3, \text{Max} \left( \frac{T(n)}{20n} \right) \) for \( n \leq 20c(a+b) \)

Prove that \( T(n) \leq 20cn \) for \( n > 20(a+b) \).
Idea:

“Two halves are better than a whole”
if the base algorithm has super-linear complexity.

“If a little's good, then more's better”
repeat above, recursively

Analysis: recursion tree or Master Recurrence

Applications: Many.

Binary Search, Merge Sort, (Quicksort), counting inversions, closest points, median, integer/ polynomial/matrix multiplication, FFT/convolution, exponentiation,...