1. (30 points) Definitions:
   - A bipartite graph is a graph in which $V = X \cup Y$, $X \cap Y = \emptyset$ and each edge has one end vertex in $X$ and one in $Y$.
   - A complete bipartite is a bipartite in which $E = X \times Y$, that is, each vertex of $X$ is connected to all the vertices of $Y$ and vice-versa.
   - A lace is a tree in which the degree of all vertices is 1 or 2 (note that this implies that a lace must be of the form $\text{o}\rightarrow\text{o}\rightarrow\ldots\rightarrow\text{o}$)
   - Reminder: the DFS-tree is a directed spanning tree in which $i \rightarrow j$ is an edge iff $i$ is the first vertex to ‘discover’ $j$ in the DFS execution.
   - An undirected graph is a lace producer if for any DFS execution (that is, for any starting vertex and any possible edge selection) the resulting DFS tree is a directed lace (i.e., of the form $\text{o}\rightarrow\text{o}\rightarrow\ldots\rightarrow\text{o}$).

For each of the following cases prove or show a counter example to the statement ‘G is a lace producer’.
1. G is a lace
2. G is a simple cycle
3. G is a complete bipartite with $|U|=|V|+1$
4. G is a complete bipartite with $|U|=|V|$
5. G consists of two simple cycles with a single joint vertex.

2. (25 points) $G=(V,E)$ is an undirected graph with weights on the edges. $s,t \in V$, $e \in E$
   Give efficient algorithms for each of the following problems:
   a. Does $e$ belong to all the shortest paths connecting $s$ and $t$?
   b. Does $e$ belong to some shortest path connecting $s$ and $t$?
   Prove that your algorithms are correct and analyze their complexity.

3. (25 points)
   a. Describe an algorithm for finding the number of shortest paths from $s$ to $t$ after the BFS algorithm has been performed. Each vertex $v$ is now labeled by $d(v)$, which is the distance of $v$ from $s$. Prove that your algorithm is correct.
   
   b. Repeat the above, after Dijkstra’s algorithm has been performed. Assume $c(e) > 0$ for every edge $e$. Why is this assumption necessary?
   You don't need to prove your algorithm.

4. (20 points) Use Bellman-Ford algorithm to suggest how we can detect in $O(|E||V|)$ steps if a directed graph contains a negative cycle. Explain briefly.