To-Do List

- Online quiz: due Sunday
- Print, sign, and return the academic integrity statement (if you haven't already)
- Read: Smith (2017); optionally, Jurafsky and Martin (2016), Collins (2011) §2, and Goldberg (2015) §§0–4, 10–13 if you want to know more about neural networks
- A1 now due April 9 (Sunday)
- Late policy: four late days
Language Models: Definitions

- $\mathcal{V}$ is a finite set of (discrete) symbols (⊇ “words” or possibly characters); $V = |\mathcal{V}|$
- $\mathcal{V}^\dagger$ is the (infinite) set of sequences of symbols from $\mathcal{V}$ whose final symbol is $\circ$
- $p : \mathcal{V}^\dagger \rightarrow \mathbb{R}$, such that:
  - For any $x \in \mathcal{V}^\dagger$, $p(x) \geq 0$
  - $\sum_{x \in \mathcal{V}^\dagger} p(X = x) = 1$

(I.e., $p$ is a proper probability distribution.)

Language modeling: estimate $p$ from examples, $\mathbf{x}_{1:n} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$.

Evaluation on test data $\mathbf{\bar{x}}_{1:m}$: perplexity, $2^{-\frac{1}{M} \sum_{i=1}^{m} \log_2 p(\mathbf{\bar{x}}_i)}$
Log-Linear Models: Definitions

We define a conditional log-linear model \( p(Y \mid X) \) as:

- \( \mathcal{Y} \) is the set of events/outputs (😊 for language modeling, \( \mathcal{Y} \))
- \( \mathcal{X} \) is the set of contexts/inputs (😊 for n-gram language modeling, \( \mathcal{V}^{n-1} \))
- \( \phi : \mathcal{X} \times \mathcal{Y} \rightarrow \mathbb{R}^d \) is a feature vector function
- \( \mathbf{w} \in \mathbb{R}^d \) are the model parameters

\[
p_{\mathbf{w}}(Y = y \mid X = x) = \frac{\exp \mathbf{w} \cdot \phi(x, y)}{\sum_{y' \in \mathcal{Y}} \exp \mathbf{w} \cdot \phi(x, y')}
\]
Breaking It Down

\[ p_w(Y = y \mid X = x) = \frac{\exp w \cdot \phi(x, y)}{\sum_{y' \in \mathcal{Y}} \exp w \cdot \phi(x, y')} \]
Breaking It Down

\[ p_w(Y = y \mid X = x) = \frac{\exp w \cdot \phi(x, y)}{\sum_{y' \in Y} \exp w \cdot \phi(x, y)} \]

linear score \( w \cdot \phi(x, y) \)
Breaking It Down

\[
p_w(Y = y \mid X = x) = \frac{\exp w \cdot \phi(x, y)}{\sum_{y' \in Y} \exp w \cdot \phi(x, y)}
\]

- linear score: \( w \cdot \phi(x, y) \)
- nonnegative: \( \exp w \cdot \phi(x, y) \)
Breaking It Down

\[ p_w(Y = y \mid X = x) = \frac{\exp \mathbf{w} \cdot \phi(x, y)}{\sum_{y' \in \mathcal{Y}} \exp \mathbf{w} \cdot \phi(x, y')} \]

- **linear score**: \( \mathbf{w} \cdot \phi(x, y) \)
- **nonnegative**: \( \exp \mathbf{w} \cdot \phi(x, y) \)
- **normalizer**: \( \sum_{y' \in \mathcal{Y}} \exp \mathbf{w} \cdot \phi(x, y') = Z_\mathbf{w}(x) \)
Breaking It Down

\[
p_w(Y = y \mid X = x) = \frac{\exp \mathbf{w} \cdot \phi(x, y)}{\sum_{y' \in Y} \exp \mathbf{w} \cdot \phi(x, y)}
\]

- **linear score** \( \mathbf{w} \cdot \phi(x, y) \)
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- **normalizer** \( \sum_{y' \in Y} \exp \mathbf{w} \cdot \phi(x, y') = Z_w(x) \)

“Log-linear” comes from the fact that:

\[
\log p_w(Y = y \mid X = x) = \mathbf{w} \cdot \phi(x, y) - \underbrace{\log Z_w(x)}_{\text{constant in } y}
\]

This is an instance of the family of **generalized linear models**.
### The Geometric View

Suppose we have instance $x$, $\mathcal{Y} = \{y_1, y_2, y_3, y_4\}$, and there are only two features, $\phi_1$ and $\phi_2$.

As a simple example, let the two features be binary functions.
The Geometric View

Suppose we have instance $x$, $\mathcal{Y} = \{y_1, y_2, y_3, y_4\}$, and there are only two features, $\phi_1$ and $\phi_2$. 

$w \cdot \phi = w_1 \phi_1 + w_2 \phi_2 = 0$
The Geometric View

Suppose we have instance $x$, $\mathcal{Y} = \{y_1, y_2, y_3, y_4\}$, and there are only two features, $\phi_1$ and $\phi_2$.

$$\text{distance}(w \cdot \phi = 0, \phi_0) = \frac{|w \cdot \phi_0|}{\|w\|_2} \propto |w \cdot \phi_0|$$
The Geometric View

Suppose we have instance $x$, $\mathcal{Y} = \{y_1, y_2, y_3, y_4\}$, and there are only two features, $\phi_1$ and $\phi_2$.

\[
\begin{align*}
\mathbf{w} \cdot \phi(x, y_1) &> \mathbf{w} \cdot \phi(x, y_3) > \mathbf{w} \cdot \phi(x, y_4) > 0 \geq \mathbf{w} \cdot \phi(x, y_2)
\end{align*}
\]
The Geometric View

Suppose we have instance $x$, $\mathcal{Y} = \{y_1, y_2, y_3, y_4\}$, and there are only two features, $\phi_1$ and $\phi_2$. 

$$p_w(y_1 \mid x) > p_w(y_3 \mid x) > p_w(y_4 \mid x) > p_w(y_2 \mid x)$$
The Geometric View

Suppose we have instance $x$, $\mathcal{Y} = \{y_1, y_2, y_3, y_4\}$, and there are only two features, $\phi_1$ and $\phi_2$. 

![Diagram showing the geometric view with points $(x, y_1)$, $(x, y_3)$, $(x, y_2)$, and $(x, y_4)$ on a 2D plane with axes $\phi_1$ and $\phi_2$.]
The Geometric View

Suppose we have instance $x$, $\mathcal{Y} = \{y_1, y_2, y_3, y_4\}$, and there are only two features, $\phi_1$ and $\phi_2$.

$p_w(y_3 \mid x) > p_w(y_1 \mid x) > p_w(y_2 \mid x) > p_w(y_4 \mid x)$
The Geometric View

Suppose we have instance $x$, $\mathcal{Y} = \{y_1, y_2, y_3, y_4\}$, and there are only two features, $\phi_1$ and $\phi_2$.

Log-linear parameter estimation tries to choose $w$ so that $p_w(Y \mid x)$ matches the empirical distribution, $\frac{c(x,Y)}{c(x)}$. 

Why Build Language Models This Way?

- Exploit features of histories for sharing of statistical strength and better smoothing (Lau et al., 1993)
- Condition the whole text on more interesting variables like the gender, age, or political affiliation of the author (Eisenstein et al., 2011)
- Interpretability!
  - Each feature $\phi_k$ controls a factor to the probability ($e^{w_k}$).
  - If $w_k < 0$ then $\phi_k$ makes the event less likely by a factor of $\frac{1}{e^{w_k}}$.
  - If $w_k > 0$ then $\phi_k$ makes the event more likely by a factor of $e^{w_k}$.
  - If $w_k = 0$ then $\phi_k$ has no effect.
Log-Linear n-Gram Models

\[
p_w(X = x) = \prod_{j=1}^{\ell} p_w(X_j = x_j \mid X_{0:j-1} = x_{0:j-1})
\]

\[
= \prod_{j=1}^{\ell} \frac{\exp w \cdot \phi(x_{0:j-1}, x_j)}{Z_w(x_{0:j-1})}
\]

assumption

\[
= \prod_{j=1}^{\ell} \frac{\exp w \cdot \phi(x_{j-n+1:j-1}, x_j)}{Z_w(x_{j-n+1:j-1})}
\]

\[
= \prod_{j=1}^{\ell} \frac{\exp w \cdot \phi(h_j, x_j)}{Z_w(h_j)}
\]
Example

The man who knew too much
many
little
few

::

hippopotamus
What Features in $\phi(X_{j-n+1:j-1}, X_j)$?
What Features in $\phi(X_{j-n+1:j-1}, X_j)$?

- Traditional n-gram features: “$X_{j-1} = \text{the} \land X_j = \text{man}$”
What Features in $\phi(X_{j-n+1:j-1}, X_j)$?

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- “Gappy” n-grams: “$X_{j-2} = \text{the} \land X_j = \text{man}$”
What Features in $\phi(X_{j-n+1:j-1}, X_j)$?

- Traditional n-gram features: “$X_{j-1} = \text{the} \land X_j = \text{man}$”
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- Spelling features: “$X_j$’s first character is capitalized”
What Features in $\phi(X_{j-n+1:j-1}, X_j)$?

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- Spelling features: “$X_j$’s first character is capitalized”
- Class features: “$X_j$ is a member of class 132”
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- Spelling features: “$X_j$’s first character is capitalized”
- Class features: “$X_j$ is a member of class 132”
- Gazetteer features: “$X_j$ is listed as a geographic place name”
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You can define any features you want!

- Too many features, and your model will overfit 😞

- Too few (good) features, and your model will not learn 😞
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You can define any features you want!

- Too many features, and your model will overfit 😊
  - “Feature selection” methods, e.g., ignoring features with very low counts, can help.
- Too few (good) features, and your model will not learn 😞
“Feature Engineering”

- Many advances in NLP (not just language modeling) have come from careful design of features.
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Sometimes “feature engineering” is used pejoratively.
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  - Some people would rather not spend their time on it!
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There is some work on automatically inducing features (Della Pietra et al., 1997).
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More recent work in neural networks can be seen as discovering features (instead of engineering them).

But in much of NLP, there’s a strong preference for interpretable features.
How to Estimate $w$?

**n-gram**

$$p_\theta(x) = \prod_{j=1}^{\ell} \theta_{x_j|h_j}$$

**Parameters:**

$$\forall v \in V, h \in (V \cup \{\bigcirc\})^{n-1}$$

**MLE:**

$$\hat{\theta}_{v|h} = \frac{c(hv)}{c(h)}$$

**log-linear n-gram**

$$\prod_{j=1}^{\ell} \exp \frac{w_k \cdot \phi(h_j, x_j)}{Z_w(h_j)}$$

**Parameters:**

$$\forall k \in \{1, \ldots, d\}$$

**MLE:**

no closed form
MLE for $w$

- Let training data consist of $\{(h_i, x_i)\}_{i=1}^N$. 

Maximum likelihood estimation is:

$$
\max_{w \in \mathbb{R}^d} \sum_{i=1}^N \log p_w(x_i | h_i) = \max_{w \in \mathbb{R}^d} \sum_{i=1}^N \log \exp \langle w, \phi(h_i, x_i) \rangle - \log \sum_{v \in V} \exp \langle w, \phi(h_i, v) \rangle Z_w(h_i).
$$

This is concave in $w$. $Z_w(h_i)$ involves a sum over $V$ terms.
MLE for $\mathbf{w}$

- Let training data consist of $\{(h_i, x_i)\}_{i=1}^N$.
- Maximum likelihood estimation is:

\[
\max_{\mathbf{w} \in \mathbb{R}^d} \sum_{i=1}^N \log p_{\mathbf{w}}(x_i \mid h_i)
\]

\[
= \max_{\mathbf{w} \in \mathbb{R}^d} \sum_{i=1}^N \log \frac{\exp \mathbf{w} \cdot \phi(h_i, x_i)}{Z_{\mathbf{w}}(h_i)}
\]

\[
= \max_{\mathbf{w} \in \mathbb{R}^d} \sum_{i=1}^N \mathbf{w} \cdot \phi(h_i, x_i) - \log \sum_{v \in \mathcal{V}} \exp \mathbf{w} \cdot \phi(h_i, v)
\]

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$Z_{\mathbf{w}}(h_i)$ involves a sum over $\mathcal{V}$ terms.
MLE for $w$

- Let training data consist of $\{(h_i, x_i)\}_{i=1}^N$.
- Maximum likelihood estimation is:

$$\max_{w \in \mathbb{R}^d} \sum_{i=1}^N \log p_w(x_i \mid h_i)$$

$$= \max_{w \in \mathbb{R}^d} \sum_{i=1}^N \log \frac{\exp w \cdot \phi(h_i, x_i)}{Z_w(h_i)}$$

$$= \max_{w \in \mathbb{R}^d} \sum_{i=1}^N w \cdot \phi(h_i, x_i) - \log \sum_{v \in \mathcal{V}} \exp w \cdot \phi(h_i, v)$$

- This is concave in $w$. 
MLE for $w$

- Let training data consist of $\{(h_i, x_i)\}_{i=1}^N$.
- Maximum likelihood estimation is:

$$\max_{w \in \mathbb{R}^d} \sum_{i=1}^N \log p_w(x_i | h_i)$$

$$= \max_{w \in \mathbb{R}^d} \sum_{i=1}^N \log \frac{\exp w \cdot \phi(h_i, x_i)}{Z_w(h_i)}$$

$$= \max_{w \in \mathbb{R}^d} \sum_{i=1}^N w \cdot \phi(h_i, x_i) - \log \sum_{v \in V} \exp w \cdot \phi(h_i, v) \underbrace{Z_w(h_i)}_{Z_w(h_i)}$$

- This is \textit{concave} in $w$.
- $Z_w(h_i)$ involves a sum over $V$ terms.
MLE for $w$

$$\max_{w \in \mathbb{R}^d} \sum_{i=1}^{N} w \cdot \phi(h_i, x_i) - \log Z_w(h_i)$$
MLE for $w$

$$\max_{w \in \mathbb{R}^d} \sum_{i=1}^{N} w \cdot \phi(h_i, x_i) - \log f_i(w)$$

Hope/fear view: for each instance $i$,

- increase the score of the correct output $x_i$, $score(x_i) = w \cdot \phi(h_i, x_i)$
- decrease the “softened max” score overall, $\log \sum_{v \in \mathcal{V}} \exp score(v)$
MLE for $w$

$$
\max_{w \in \mathbb{R}^d} \sum_{i=1}^{N} \left\{ w \cdot \phi(h_i, x_i) - \log Z_w(h_i) \right\}
$$

Gradient view:

$$
\nabla_w f_i = \phi(h_i, x_i) - \sum_{v \in \mathcal{V}} p_w(v \mid h_i) \cdot \phi(h_i, v)
$$

observed features

expected features

Setting this to zero means getting model’s expectations to match empirical observations.
MLE for $w$: Algorithms

- Batch methods (L-BFGS is popular)
- Stochastic gradient ascent/descent more common today, especially with special tricks for adapting the step size over time
- Many specialized methods (e.g., “iterative scaling”)

Stochastic Gradient Descent

Goal: minimize $\sum_{i=1}^{N} f_i(w)$ with respect to $w$.

Input: initial value $w$, number of epochs $T$, learning rate $\alpha$

For $t \in \{1, \ldots, T\}$:

- Choose a random permutation $\pi$ of $\{1, \ldots, N\}$.
- For $i \in \{1, \ldots, N\}$:

$$w \leftarrow w - \alpha \cdot \nabla_w f_{\pi(i)}$$

Output: $w$
Avoiding Overfitting

Maximum likelihood estimation:

$$\max_{\mathbf{w} \in \mathbb{R}^d} \sum_{i=1}^{N} \mathbf{w} \cdot \phi(h_i, x_i) - \log Z_\mathbf{w}(h_i)$$

- If $\phi_j(h, x)$ is (almost) always positive, we can always increase the objective (a little bit) by increasing $w_j$ toward $+\infty$. 


Avoiding Overfitting

Maximum likelihood estimation:

$$\max_{\mathbf{w} \in \mathbb{R}^d} \sum_{i=1}^{N} \mathbf{w} \cdot \phi(h_i, x_i) - \log Z_{\mathbf{w}}(h_i)$$

- If $\phi_j(h, x)$ is (almost) always positive, we can always increase the objective (a little bit) by increasing $w_j$ toward $+\infty$.

Standard solution is to add a regularization term:

$$\max_{\mathbf{w} \in \mathbb{R}^d} \sum_{i=1}^{N} \mathbf{w} \cdot \phi(h_i, x_i) - \log \sum_{v \in \mathcal{V}} \exp \mathbf{w} \cdot \phi(h_i, v) - \lambda \|\mathbf{w}\|^p_p$$

where $\lambda > 0$ is a hyperparameter and $p = 2$ or $1$. 
MLE for \( w \)

If we had more time, we’d study this problem more carefully!

Here’s what you must remember:

- There is no closed form; you must use a numerical optimization algorithm like stochastic gradient descent.
- Log-linear models are powerful but expensive (\( Z_w(h_i) \)).
- Regularization is very important; we don’t actually do MLE.
  - Just like for n-gram models! Only even more so, since log-linear models are even more expressive.
Quick Recap

Two kinds of language models so far:

<table>
<thead>
<tr>
<th>representation?</th>
<th>estimation?</th>
<th>think about?</th>
</tr>
</thead>
<tbody>
<tr>
<td>n-gram</td>
<td>count and normalize</td>
<td>smoothing</td>
</tr>
<tr>
<td>( h_i ) is ( (n - 1) ) previous symbols</td>
<td>iterative gradient descent</td>
<td>features</td>
</tr>
<tr>
<td>log-linear</td>
<td>featurized representation of ( \langle h_i, x_i \rangle )</td>
<td>features</td>
</tr>
</tbody>
</table>
A feedforward neural network $n_\nu$ is defined by:
- A function family that maps parameter values to functions of the form $n : \mathbb{R}^{d_{in}} \rightarrow \mathbb{R}^{d_{out}}$; typically:
  - Non-linear
  - Differentiable with respect to its inputs
  - “Assembled” through a series of affine transformations and non-linearities, composed together
  - Symbolic/discrete inputs handled through lookups.
- Parameter values $\nu$
  - Typically a collection of scalars, vectors, and matrices
  - We often assume they are linearized into $\mathbb{R}^D$
A Couple of Useful Functions

- **softmax**: $\mathbb{R}^k \rightarrow \mathbb{R}^k$

  \[ \langle x_1, x_2, \ldots, x_k \rangle \mapsto \left( \frac{e^{x_1}}{\sum_{j=1}^k e^{x_j}}, \frac{e^{x_2}}{\sum_{j=1}^k e^{x_j}}, \ldots, \frac{e^{x_k}}{\sum_{j=1}^k e^{x_j}} \right) \]

- **tanh**: $\mathbb{R} \rightarrow [-1, 1]$

  \[ x \mapsto \frac{e^x - e^{-x}}{e^x + e^{-x}} \]

  Generalized to be *elementwise*, so that it maps $\mathbb{R}^k \rightarrow [-1, 1]^k$.

- Others include: ReLUs, logistic sigmoids, PReLU, etc.
“One Hot” Vectors

Arbitrarily order the words in $\mathcal{V}$, giving each an index in $\{1, \ldots, V\}$.

Let $e_i \in \mathbb{R}^V$ contain all zeros, with the exception of a 1 in position $i$.

This is the “one hot” vector for the $i$th word in $\mathcal{V}$.
Feedforward Neural Network Language Model
(Bengio et al., 2003)

Define the n-gram probability as follows:

\[
p(\cdot \mid \langle h_1, \ldots, h_{n-1} \rangle) = \nu \left( \langle e_{h_1}, \ldots, e_{h_{n-1}} \rangle \right) =
\]

\[
\text{softmax} \left( b + \sum_{j=1}^{n-1} e_{h_j}^T M A_j + W \tanh \left( u + \sum_{j=1}^{n-1} e_{h_j}^T M T_j \right) \right)
\]

where each \( e_{h_j} \in \mathbb{R}^V \) is a one-hot vector and \( H \) is the number of “hidden units” in the neural network (a “hyperparameter”).

Parameters \( \nu \) include:

- \( M \in \mathbb{R}^{V \times d} \), which are called “embeddings” (row vectors), one for every word in \( V \)
- Feedforward NN parameters \( b \in \mathbb{R}^V \), \( A \in \mathbb{R}^{(n-1) \times d \times V} \), \( W \in \mathbb{R}^{V \times H} \), \( u \in \mathbb{R}^H \), \( T \in \mathbb{R}^{(n-1) \times d \times H} \)
Breaking It Down

Look up each of the history words $h_j, \forall j \in \{1, \ldots, n - 1\}$ in $M$; keep two copies.

$$e_{h_j}^T M_{v \times d}$$
$$e_{h_j}^T M_{v \times d}$$
Breaking It Down

Look up each of the history words $h_j, \forall j \in \{1, \ldots, n-1\} \text{ in } \mathbf{M}$; keep two copies. Rename the embedding for $h_j$ as $\mathbf{m}_{h_j}$.

\[
\mathbf{e}_{h_j}^\top \mathbf{M} = \mathbf{m}_{h_j} \\
\mathbf{e}_{h_j}^\top \mathbf{M} = \mathbf{m}_{h_j}
\]
Breaking It Down

Apply an affine transformation to the second copy of the history-word embeddings \((u, T)\)

\[
\mathbf{u} + \sum_{j=1}^{n-1} \mathbf{m}_{h_j} \mathbf{T}_j^{d \times H}
\]
Apply an affine transformation to the second copy of the history-word embeddings \((u, T)\) and a \(\tanh\) nonlinearity.

\[
\begin{align*}
\mathbf{m}_{h_j} \\
\tanh \left( u + \sum_{j=1}^{n-1} \mathbf{m}_{h_j} T_j \right)
\end{align*}
\]
Apply an affine transformation to everything \((b, A, W)\).

\[
\begin{align*}
\mathbf{b}_v + \sum_{j=1}^{n-1} \mathbf{m}_{h_j} \mathbf{A}_j \\
+ \mathbf{W}_{v \times H} \tanh \left( \mathbf{u} + \sum_{j=1}^{n-1} \mathbf{m}_{h_j} \mathbf{T}_j \right)
\end{align*}
\]
Breaking It Down

Apply a softmax transformation to make the vector sum to one.

\[
\text{softmax} \left( b + \sum_{j=1}^{n-1} m_{h_j} A_j \right) + W \text{tanh} \left( u + \sum_{j=1}^{n-1} m_{h_j} T_j \right)
\]
Breaking It Down

$$\text{softmax} \left( b + \sum_{j=1}^{n-1} m_{h_j} A_j \right)$$

$$+ W \tanh \left( u + \sum_{j=1}^{n-1} m_{h_j} T_j \right)$$

Like a log-linear language model with two kinds of features:

- Concatenation of context-word embeddings vectors $m_{h_j}$
- $\tanh$-affine transformation of the above

New parameters arise from (i) embeddings and (ii) affine transformation “inside” the nonlinearity.
Number of Parameters

\[ D = V_d + V_b + (n - 1)dV + V_H + H_u + (n - 1)dH \]

For Bengio et al. (2003):
▶ \( V \approx 18000 \) (after OOV processing)
▶ \( d \in \{30, 60\} \)
▶ \( H \in \{50, 100\} \)
▶ \( n - 1 = 5 \)

So \( D = 461V + 30100 \) parameters, compared to \( O(V^n) \) for classical n-gram models.
▶ Forcing \( A = 0 \) eliminated 300V parameters and performed a bit better, but was slower to converge.
▶ If we averaged \( m_{h_j} \) instead of concatenating, we’d get to \( 221V + 6100 \) (this is a variant of “continuous bag of words,” Mikolov et al., 2013).
Why does it work?

Historical answer: multiple layers and nonlinearities allow feature combinations a linear model can't get.

Suppose we want \( y = \text{xor}(x_1, x_2) \); this can't be expressed as a linear function of \( x_1 \) and \( x_2 \).

With high-dimensional inputs, there are a lot of conjunctive features to search through. For log-linear models, Della Pietra et al. (1997) did this, greedily.

Neural models seem to smoothly explore lots of approximately-conjunctive features.

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Tuples where $y = \text{xor}(x_1, x_2)$ are red; tuples where $y \neq \text{xor}(x_1, x_2)$ are blue.
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  - Suppose we want \( y = \text{xor}(x_1, x_2); \) this can’t be expressed as a linear function of \( x_1 \) and \( x_2 \). But:

\[
  z = x_1 \cdot x_2 \\
y = x_1 + x_2 - 2z
\]

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xor Example ($D = 13$)

Credit: Chris Dyer (https://github.com/clab/cnn/blob/master/examples/xor.cc)

\[
\min_{v,a,W,b} \sum_{x_1 \in \{0,1\}} \sum_{x_2 \in \{0,1\}} \left( \text{xor}(x_1, x_2) - v^\top (W x + b)^3 + a \right)^2
\]

\[
\min_{v,a,W,b} \sum_{x_1 \in \{0,1\}} \sum_{x_2 \in \{0,1\}} \left( \text{xor}(x_1, x_2) - v^\top \tanh(W x + b)^3 + a \right)^2
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- Word embeddings: a powerful idea . . .
Important Idea: Words as Vectors

The idea of “embedding” words in $\mathbb{R}^d$ is much older than neural language models.
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The idea of “embedding” words in $\mathbb{R}^d$ is much older than neural language models. You should think of this as a generalization of the discrete view of $\mathcal{V}$. 

Considerable ongoing research on learning word representations to capture linguistic similarity (Turney and Pantel, 2010); this is known as vector space semantics.
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Words as Vectors: Example

baby

cat
Words as Vectors: Example

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pig
mouse
cat
Bad news for neural language models:

- Log-likelihood function is not concave.
  - So any perplexity experiment is evaluating the model and an algorithm for estimating it.
- Calculating log-likelihood and its gradient is very expensive (5 epochs took 3 weeks on 40 CPUs).
Parameter Estimation

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Good news:
- $n_\nu$ is differentiable with respect to $M$ (from which its inputs come) and $\nu$ (its parameters), so gradient-based methods are available.
  - Essential: the chain rule from calculus (sometimes called “backpropagation”)

Lots more details in Bengio et al. (2003) and (for NNs more generally) in Goldberg (2015).
More examples of neural language models (in brief):

- The log-bilinear language model
- Recurrent neural network language models
Log-Bilinear Language Model
(Mnih and Hinton, 2007)

Define the n-gram probability as follows, for each \( v \in \mathcal{V} \):

\[
p(v \mid \langle h_1, \ldots, h_{n-1} \rangle) = \frac{\exp \left( \sum_{j=1}^{n-1} \left( \mathbf{m}_{h_j}^T \mathbf{A}_j + \mathbf{b}^T \right) \mathbf{m}_v + c_v \right)}{\sum_{v' \in \mathcal{V}} \exp \left( \sum_{j=1}^{n-1} \left( \mathbf{m}_{h_j}^T \mathbf{A}_j + \mathbf{b}^T \right) \mathbf{m}_{v'} + c_v \right)}
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- Number of parameters: $D = Vd + \frac{(n-1)d^2}{M} + \frac{d}{A} + V$
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- The predicted word’s probability depends on its vector \( \mathbf{m}_v \), not just on the vectors of the history words.
- Training this model involves a sum over the vocabulary (like log-linear models we saw earlier).
- Later work explored variations to make learning faster.
Observations about Neural Language Models (So Far)

- There’s no knowledge built in that the most recent word $h_{n-1}$ should generally be more informative than earlier ones.
  - This has to be learned.
- In addition to choosing $n$, also have to choose dimensionalities like $d$ and $H$.
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- Parameters of these models are hard to interpret.
  - Example: $\ell_2$-norm of $A_j$ and $T_j$ in the feedforward model correspond to the importance of history position $j$.
  - Individual word embeddings can be clustered and dimensions can be analyzed (e.g., Tsvetkov et al., 2015).
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- In addition to choosing $n$, also have to choose dimensionalities like $d$ and $H$.
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- Architectures are not intuitive.
- Still, impressive perplexity gains got people’s interest.
Recurrent Neural Network

- Each input element is understood to be an element of a sequence: \( \langle x_1, x_2, \ldots, x_\ell \rangle \)
- At each timestep \( t \):
  - The \( t \)th input element \( x_t \) is processed alongside the previous state \( s_{t-1} \) to calculate the new state \( s_t \).
  - The \( t \)th output is a function of the state \( s_t \).
  - The same functions are applied at each iteration:
    \[
    s_t = f_{\text{recurrent}}(x_t, s_{t-1}) \\
    y_t = f_{\text{output}}(s_t)
    \]

In RNN language models, words and histories are represented as vectors (respectively, \( x_t = e_{x_t} \) and \( s_t \)).
RNN Language Model

The original version, by Mikolov et al. (2010) used a “simple” RNN architecture along these lines:

\[ s_t = f_{\text{recurrent}}(e_{x_t}, s_{t-1}) = \text{sigmoid} \left( (e_{x_t}^T M)^T A + s_{t-1}^T B + c \right) \]

\[ y_t = f_{\text{output}}(s_t) = \text{softmax} \left( s_t^T U \right) \]

\[ p(v \mid x_1, \ldots, x_{t-1}) = [y_t]_v \]

Note: this is not an n-gram (Markov) model!
Visualization
Improvements to RNN Language Models

The simple RNN is known to suffer from two related problems:

- “Vanishing gradients” during learning make it hard to propagate error into the distant past.
- State tends to change a lot on each iteration; the model “forgets” too much.

Some variants:

- “Stacking” these functions to make deeper networks.
- Sundermeyer et al. (2012) use “long short-term memories” (LSTMs) and Cho et al. (2014) use “gated recurrent units” (GRUs) to define $f_{\text{recurrent}}$.
- Mikolov et al. (2014) engineer the linear transformation in the simple RNN for better preservation.
- Jozefowicz et al. (2015) used randomized search to find even better architectures.
## Comparison: Probabilistic vs. Connectionist Modeling

<table>
<thead>
<tr>
<th></th>
<th>Probabilistic</th>
<th>Connectionist</th>
</tr>
</thead>
<tbody>
<tr>
<td>What do we engineer?</td>
<td>features, assumptions</td>
<td>architectures</td>
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<tr>
<td>Theory?</td>
<td>as $N$ gets large</td>
<td>not really</td>
</tr>
<tr>
<td>Interpretation of parameters?</td>
<td>often easy</td>
<td>usually hard</td>
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Parting Shots

I said very little about estimating the parameters. At present, this requires a lot of engineering. New libraries to help you are coming out all the time. Many of them use GPUs to speed things up. This progression is worth reflecting on: history: represented as:

before 1996 (n−1)-gram discrete
1996–2003 feature vector
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