CSEP505: Programming Languages Lecture 4: Untyped Lambda-Calculus, Formal Operational Semantics,

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Where are we

- To talk about functions more precisely, we need to define them as carefully as we did IMP's constructs
- First try adding functions & local variables to IMP "on the cheap"
 It didn't work [see last week]
- Now back up and define a language with *nothing* but functions
 - [started last week]
 - And then *encode* everything else

Review

- Cannot properly model local scope via a global heap of integers
 - Functions are not syntactic sugar for assignments to globals
- So let's build a model of this key concept
 - Or just borrow one from 1930s logic
- And for now, drop mutation, conditionals, and loops
 - We won't need them!
- The Lambda calculus in BNF

Expressions: $e ::= x | \lambda x. e | e e$ Values: $v ::= \lambda x. e$

That's all of it! [More review]

Expressions: $e ::= x | \lambda x. e | e e$ Values: $v ::= \lambda x. e$

A program is an *e*. To call a function:

substitute the argument for the bound variable

That's the key operation we were missing

Example substitutions:

$$(\lambda x. x) (\lambda y. y) \rightarrow \lambda y. y$$

 $(\lambda x. \lambda y. y x) (\lambda z. z) \rightarrow \lambda y. y (\lambda z. z)$
 $(\lambda x. x x) (\lambda x. x x) \rightarrow (\lambda x. x x) (\lambda x. x x)$

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Why substitution [More review]

- After substitution, the bound variable is *gone*
 - So clearly its name didn't matter
 - That was our problem before
- Given substitution we can define a little programming language
 - (correct & precise definition is subtle; we'll come back to it)
 - This microscopic PL turns out to be Turing-complete

Full large-step interpreter

```
type exp = Var of string
         | Lam of string*exp
         | Apply of exp * exp
exception BadExp
let subst e1 with e2 for x = ...(*to be discussed*)
let rec interp large e =
 match e with
  Var -> raise BadExp(* unbound variable *)
 Lam -> e (* functions are values *)
 Apply(e1,e2) ->
    let v1 = interp large e1 in
    let v2 = interp_large e2 in
   match v1 with
      Lam(x,e3) -> interp large (subst e3 v2 x)
    | -> failwith "impossible" (* why? *)
```

Interpreter summarized

- Evaluation produces a value Lam(x,e3) if it terminates
- Evaluate application (call) by
 - 1. Evaluate left
 - 2. Evaluate right
 - 3. Substitute result of (2) in body of result of (1)
 - 4. Evaluate result of (3)

A different semantics has a different *evaluation strategy*.

- 1. Evaluate left
- 2. Substitute right in body of result of (1)
- 3. Evaluate result of (2)

Another interpreter

```
type exp = Var of string
         | Lam of string*exp
         | Apply of exp * exp
exception BadExp
let subst e1 with e2 for x = ...(*to be discussed*)
let rec interp large2 e =
 match e with
  Var -> raise BadExp(*unbound variable*)
 Lam -> e (*functions are values*)
 Apply(e1,e2) ->
    let v1 = interp large2 e1 in
    (* we used to evaluate e2 to v2 here *)
   match v1 with
      Lam(x,e3) -> interp_large2 (subst e3 e2 x)
    | -> failwith "impossible" (* why? *)
```

What have we done

- Syntax and two large-step semantics for the untyped lambda calculus
 - First was "call by value"
 - Second was "call by name"
- Real implementations don't use substitution
 - They do something *equivalent*
- Amazing (?) fact:
 - If call-by-value terminates, then call-by-name terminates
 - (They might both not terminate)

What will we do

- Go back to math metalanguage
 - Notes on concrete syntax (relates to OCaml)
 - Define semantics with inference rules
- Lambda encodings (show our language is mighty)
- Define substitution precisely
- Environments

Next time??

- Small-step
- Play with *continuations* ("very fancy" language feature)

Syntax notes

- When in doubt, put in parentheses
- Math (and OCaml) resolve ambiguities as follows:
- 1. $\lambda x. e1 e2 is (\lambda x. e1 e2)$
 - not (λx. e1) e2

General rule: Function body "starts at the dot" and "ends at the first unmatched right paren"

Example:

 $(\lambda x. y (\lambda z. z) w) q$

Syntax notes

- 2. e1 e2 e3 is (e1 e2) e3
 - not e1 (e2 e3)

General rule: Application "associates to the left"

So e1 e2 e3 e4 is (((e1 e2) e3) e4)

It's just syntax

- As in IMP, we really care about abstract syntax
 - Here, internal tree nodes labeled " λ " or "apply" (i.e., "call")
- Previous 2 rules just reduce parens when writing trees as strings
- Rules may seem strange, but they're the most convenient
 - Based on 70 years experience
 - Especially with currying

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Inference rules

- A metalanguage for operational semantics
 - Plus: more concise (& readable?) than OCaml
 - Plus: useful for reading research papers
 - Plus: natural support for nondeterminism
 - Definition allowing observably different implementations
 - Minus: less tool support than OCaml (no compiler)
 - Minus: one more thing to learn
 - Minus: painful in Powerpoint

Informal idea

Want to know:

what values (0, 1, many?) an expression can evaluate to

So define a *relation* over *pairs* (**e**,**v**):

- Where \mathbf{e} is an expression and \mathbf{v} is a value
- Just a subset of all pairs of expressions and values

If the language is deterministic, this *relation* turns out to be a *function* from expressions to values

Metalanguage supports defining relations

- Then prove a relation is a function (if it is)

Making up metasyntax

Rather than write (\mathbf{e}, \mathbf{v}) , we'll write $e \mathbf{\Psi} v$.

- It's just metasyntax (!)
 - Could use interp(e,v) or « v e » if you prefer
- Our metasyntax follows PL convention
 - Colors are not conventional (slides: green = metasyntax)
- And distinguish it from other relations

First step: define the *form* (arity and metasyntax) of your relation(s):



This is called a *judgment*

What we need to define

So we can write $e \checkmark v$ for any e and v

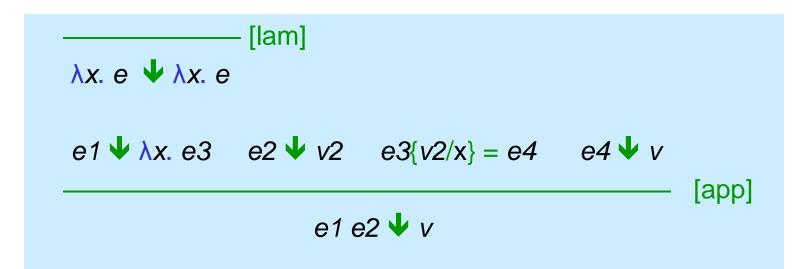
 But we want such a thing to be "true" to mean e can evaluate to v and "false" to mean it cannot

Examples (before the definition):

- $(\lambda x. \lambda y. y x) ((\lambda z. z) (\lambda z. z)) \checkmark \lambda y. y (\lambda z. z)$ in the relation
- $(\lambda x. \lambda y. y x) ((\lambda z. z) (\lambda z. z)) \checkmark \lambda z. z$ not in the relation
- $-\lambda y. y \mathbf{i} \lambda y. y$ in the relation
- $(\lambda y. y) (\lambda x. \lambda y. y x) \checkmark \lambda y. y$ not in the relation
- $(\lambda x. x x) (\lambda x. x x) \checkmark \lambda y. y$ not in the relation
- $(\lambda x. x x) (\lambda x. x x) \Psi (\lambda x. x x) (\lambda x. x x)$ metasyntactically bogus

Inference rules

$$e \mathbf{v} \vee \mathbf{v} = \mathbf{e}'$$

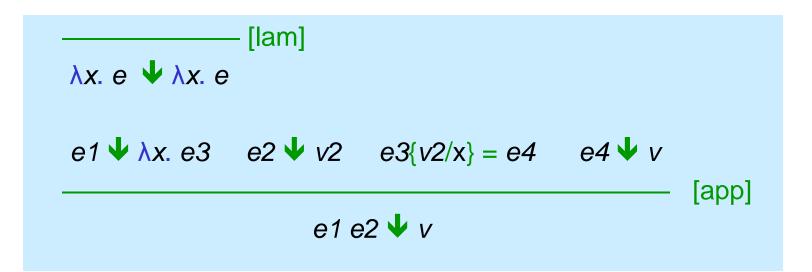


- Using definition of a set of 4-tuples for substitution
 - (exp * value * variable * exp)
 - Will define substitution later

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Inference rules

$$e \Psi V$$
 $e\{v/x\} = e'$



- Rule top: *hypotheses* (0 or more)
- Rule bottom: conclusion
- Metasemantics: If all hypotheses hold, then conclusion holds

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Rule schemas

$$e1 \checkmark \lambda x. e3 \quad e2 \checkmark v2 \quad e3\{v2/x\} = e4 \quad e4 \checkmark v$$

$$e1 e2 \checkmark v$$
[app]

- Each rule is a schema you "instantiate consistently"
- So [app] "works" "for all" x, e1, e2, e3, e4, v2, and v
- But "each" e1 has to be the "same" expression
 - Replace metavariables with appropriate terms
 - Deep connection to logic programming (e.g., Prolog)

Instantiating rules

- Two example legitimate instantiations:
 - λz. z 🔸 λz. z
 - x instantiated with z, e instantiated with z
 - λz. λy. y z ↓ λz. λy. y z
 - x instantiated with z, e instantiated with λy . y z
- Two example illegitimate instantiations:
 - λz. z ↓ λy. z
 - λz. λy. y z ↓ λz. λz. Ζ

Must get your rules "just right" so you don't allow too much or too little

Derivations

- Tuple is "in the relation" if there exists a derivation of it
 - An upside-down (or not?!) tree where each node is an instantiation and leaves are axioms (no hypotheses)
- To show $e \mathbf{v}$ for some e and v, give a derivation
 - But we rarely "hand-evaluate" like this
 - We're just defining a semantics remember
- Let's work through an example derivation for (λx. λy. y x) ((λz. z) (λz. z)) ↓ λy. y (λz. z)

Which relation?

So exactly which relation did we define

- The pairs at the *bottom of finite-height derivations*

Note: A derivation tree is like the tree of calls in a large-step interpreter

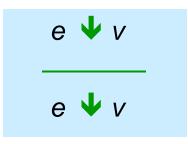
- [when relation is a function]
- Rule being instantiated is branch of the match-expression
- Instantiation is arguments/results of the recursive call

A couple extremes

 This rules are a *bad idea* because either one adds all pairs to the relation



• This rule is *pointless* because it adds no pairs to the relation



Summary so far

- Define judgment via a collection of inference rules
 - Tuple in the relation ("judgment holds") if a derivation (tree of instantiations ending in axioms) exists

As an interpreter, could be "nondeterministic":

- Multiple derivations, maybe multiple v such that $e \Psi v$
 - Our example language is deterministic
 - In fact, "syntax directed" (≤1 rule per syntax form)
- Still need rules for $e\{v/x\}=e'$
- Let's do more judgments (i.e., languages) to get the hang of it...

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Call-by-name large-step

$$e \Psi_{N} v \qquad e\{v/x\} = e'$$
[lam]

$$\lambda x. e \Psi_{N} \lambda x. e$$

$$e1 \Psi_{N} \lambda x. e3 \qquad e3\{e2/x\} = e4 \qquad e4 \Psi_{N} v$$
[app]

$$e1 e2 \Psi_{N} v$$

- Easier to see the difference than in OCaml
- Formal statement of amazing fact:
 For all *e*, if there exists a *v* such that *e* ↓ *v* then there exists a *v*2 such that *e* ↓_N *v*2

(Proof is non-trivial & must reason about substitution)

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IMP

- Two judgments $H; e \mathbf{\downarrow} i$ and $H; s \mathbf{\downarrow} H2$
- Assume get(H, x, i) and set(H, x, i, H2) are defined
- Let's try writing out inference rules for the judgments...

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Encoding motivation

- Fairly crazy: we left out integers, conditionals, data structures, ...
- Turns out we're Turing complete
 - We can encode whatever we need
 - (Just like assembly language can)
- Motivation for encodings
 - Fun and mind-expanding
 - Shows we are not oversimplifying the model ("numbers are syntactic sugar")
 - Can show languages are too expressive
 Example: C++ template instantiation
- Encodings are also just "(re)definition via translation"

Lecture 4

Encoding booleans

The "Boolean Abstract Data Type (ADT)"

- There are 2 booleans and 1 conditional expression
 - The conditional takes 3 (curried) arguments
 - If 1st argument is one bool, return 2nd argument
 - If 1st argument is other bool, return 3rd argument
- Any set of 3 expressions meeting this specification is a proper encoding of booleans
- Here is one (of many):
 - "true" λx . λy . x
 - "false" λx . λy . y
 - "if" $\lambda b. \lambda t. \lambda f. b t f$

Example

- Given our encoding:
 - "true" λx . λy . x
 - "false" λx . λy . y
 - "if" $\lambda b. \lambda t. \lambda f. b t f$
- We can derive "if" "true" v1 v2 ↓ v1
- And every "law of booleans" works out
 And every non-law does not
- By the way, this is OOP

But...

- Evaluation order matters!
 - With ↓, our "if" is not YFL's if

"if" "true" (λx. x) (λx. x x) (λx. x x) doesn't terminate but
 "if" "true" (λx. x) (λx. x x) (λx. x x) z) terminates

– Such "thunking" is unnecessary using Ψ_N

Encoding pairs

- The "Pair ADT"
 - There is 1 constructor and 2 selectors
 - 1st selector returns 1st argument passed to the constructor
 - 2nd selector returns 2nd argument passed to the constructor
- This does the trick:
 - "make_pair" λx . λy . λz . $z \times y$
 - "first" $\lambda p. p (\lambda x. \lambda y. x)$
 - "second" $\lambda p. p (\lambda x. \lambda y. y)$
- Example:

```
"snd" ("fst" ("make_pair" ("make_pair" v1 v2) v3)) \Psi v2
```

Reusing Lambda

- Is it weird that the encodings of Booleans and pairs both used
 (λx. λy. x) and (λx. λy. y) for different purposes?
- Is it weird that the same bit-pattern in binary code can represent an int, a float, an instruction, or a pointer?
- Von Neumann: Bits can represent (all) code and data
- Church (?): Lambdas can represent (all) code and data
- Beware the "Turing tarpit"

Encoding lists

- Why start from scratch? Can build on bools and pairs:
 - "empty-list" is "make_pair" "false" "false"
 - "cons" is $\lambda h. \lambda t.$ "make_pair" "true" "make_pair" h t
 - "is-empty" is ...
 - "head" is ...
 - "tail" is ...
- Note:
 - Not too far from how lists are implemented
 - Taking "tail" ("tail" "empty") will produce some lambda
 - Just like, without page-protection hardware,
 - null->tail->tail would produce some bit-pattern

Encoding natural numbers

- Known as "Church numerals"
 - Will skip in the interest of time
- The "natural number" ADT is basically:
 - "zero"
 - "successor" (the add-one function)
 - "plus"
 - "is-equal"
- Encoding is correct if "is-equal" agrees with elementary-school arithmetic
- [Don't need "full" recursion, but with "full" recursion, can also just do lists of Booleans...]

Lecture 4

Recursion

• Can we write *useful* loops? Yes!

To write a recursive function:

- Write a function that takes an *f* and call *f* in place of recursion:
 - Example (in enriched language):

 $\lambda f. \lambda x.$ if x=0 then 1 else (x * f(x-1))

- Then apply "fix" to it to get a recursive function "fix" $\lambda f. \lambda x.$ if x=0 then 1 else (x * f(x-1))
- Details, especially in CBV are icky; but it's possible and need be done only once. *For the curious:*

"fix" is $\lambda f. (\lambda x. f (\lambda y. x x y)) (\lambda x. f (\lambda y. x x y))$

More on "fix"

- "fix" is also known as the Y-combinator
- The informal idea:

- "fix" (\lambda f.e) becomes something like

e{("fix" (λf.e)) / f}

- That's unrolling the recursion once
- Further unrollings are delayed (happen as necessary)
- Teaser: Most type systems disallow "fix"
 - So later we'll add it as a primitive
 - Example: OCaml can never type-check (x x)

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Our goal

Need to define

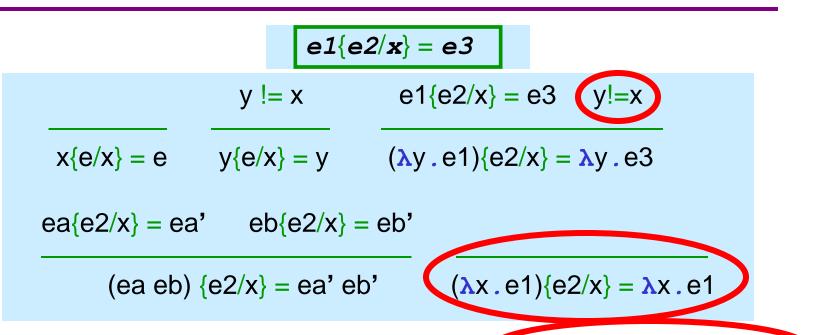
- Used in [app] rule
- Informally, "replace occurrences of x in e1 with e2"
- Shockingly subtle to get right (in theory or programming)
- (Under call-by-value, only need e2 to be a value, but that doesn't make it much easier, so define the more general thing.)

Try #1[WRONG]

	y != x	$e1{e2/x} = e3$			
$x{e/x} = e$	$y{e/x} = y$	$(\lambda y.e1){e2/x} = \lambda y.e3$			
ea{e2/2	x} = ea'	$eb{e2/x} = eb'$			
$(ea eb) \{e2/x\} = ea' eb'$					

- Recursively replace every x leaf with e2
- But the rule for substituting into (nested) functions is wrong: If the function's argument binds the same variable (shadowing), we should not change the function's body
- Example program: $(\lambda x. \lambda x. x)$ 42

Try #2 [WRONG]



- Recursively replace every x leaf with e2, but respect shadowing
- Still wrong due to capture [actual technical term]:
 - Example: $(\lambda y.e1){y/x}$
 - Example $(\lambda y.e1){(\lambda z.y/x)}$
 - In general, if "y appears free in e2"

More on capture

- Good news: capture can't happen under CBV or CBN
 If program starts with no unbound ("free") variables
- Bad news: Can still result from "inlining"
- Bad news: It's still "the wrong definition" in general
 - My experience: The nastiest of bugs in language tools

Try #3 [Almost Correct]

- First define an expression's "free variables" (braces here are set notation)
 - $FV(x) = \{x\}$
 - FV(e1 e2) = FV(e1) U FV(e2)
 - $FV(\lambda y.e) = FV(e) \{y\}$
- Now require "no capture":

$$e1\{e2/x\} = e3 \quad y!=x \quad y \text{ not in FV(e2)}$$
$$(\lambda y . e1)\{e2/x\} = \lambda y . e3$$

Try #3 in Full

	e	1{e2/x} = e3	
	y != x	$e1{e2/x} = e3$	y!=x y not in FV(e2)
$x{e/x} = e$	$y{e/x} = y$	(λ y.e1){	$\{e2/x\} = \lambda y \cdot e3$
$ea\{e2/x\} = ea'$	$eb{e2/x} =$	eb'	
(ea eb) {e2/x} = ea' eb'		' (λx.e1){e	$2/x\} = \lambda x . e1$

- No mistakes with what is here...
- ... but only a partial definition
 - What if y is in the free-variables of e2

Implicit renaming

$$e1{e2/x} = e3$$
 y!=x y not in FV(e2)

$$(\lambda y.e1){e2/x} = \lambda y.e3$$

- But this is a partial definition due to a "syntactic accident", until...
- We allow "implicit, systematic renaming" of any term
 - In general, we never distinguish terms that differ only in variable names
 - A key language-design principle
 - Actual variable choices just as "ignored" as parens
 - Means rule above can "always apply" with a lambda
- Called "alpha-equivalence": terms differing only in names of variables are the same term

Lecture 4

Try #4 [correct]

• [Includes systematic renaming and drops an unneeded rule]

	y != x	$e1{e2/x} = e3$	y!=x	y not in FV(e2)		
$x{e/x} = e$	$y{e/x} = y$	= y $(\lambda y . e1){e2/x} = \lambda y . e3$				
$ea\{e2/x\} = ea'$ $eb\{e2/x\} = eb'$						
(ea eb) {e2/x} = ea' eb'		(<u>x</u> .e1){e2	$2/x\} = \lambda$	x.et		

More explicit approach

- While "everyone in the PL field":
 - Understands the capture problem
 - Avoids it by saying "implicit systematic renaming"
 you may find that unsatisfying...
 - ... especially if you have to implement substitution while avoiding capture
- So this more explicit version also works ("fresh z for y"):

z not in FV(e1) U FV(e2) U {x} $e1{z/y} = e3 e3{e2/x} = e4$

 $(\lambda y.e1){e2/x} = \lambda z.e4$

 You have to "find an appropriate z", but one always exists and \$\$tmp appended to a global counter "probably works"

Note on metasyntax

- Substitution often thought of as a metafunction, not a judgment
 - I've seen many nondeterministic languages
 - But never a nondeterministic definition of substitution
- So instead of writing:

 $e1 \checkmark \lambda x. e3 \quad e2 \checkmark v2 \quad e3\{v2/x\} = e4 \quad e4 \checkmark v$ $e1 e2 \checkmark v$ [app]

• Just write:

$$e1 \checkmark \lambda x. e3 \quad e2 \checkmark v2 \quad e3\{v2/x\} \checkmark v$$

$$e1 e2 \checkmark v$$
[app]

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Where we're going

- Done: large-step for untyped lambda-calculus
 - CBV and CBN
 - Note: infinite number of other "reduction strategies"
 - Amazing fact: all equivalent if you ignore termination!
- Now other semantics, all equivalent to CBV:
 - With environments (in OCaml to prep for Homework 3)
 - Basic small-step (easy)
 - Contextual semantics (similar to small-step)
 - Leads to precise definition of *continuations*

Slide repeat...

```
type exp = Var of string
         | Lam of string*exp
         | Apply of exp * exp
exception BadExp
let subst e1 with e2 for x = ...(*to be discussed*)
let rec interp large e =
 match e with
  Var -> raise BadExp(*unbound variable*)
 Lam -> e (*functions are values*)
 Apply(e1,e2) ->
    let v1 = interp large e1 in
    let v_2 = interp large e2 in
   match v1 with
      Lam(x,e3) -> interp large (subst e3 v2 x)
    | -> failwith "impossible" (* why? *)
```

Environments

- Rather than substitute, let's keep a map from variables to values
 - Called an environment
 - Like IMP's heap, but immutable and 1 not enough
- So a program "state" is now exp and environment
- A function body is evaluated under the environment where it was defined!
 - Use closures to store the environment
 - See also Lecture 1

No more substitution

```
type exp = Var of string
         | Lam of string * exp
         | Apply of exp * exp
         | Closure of string * exp * env
and env = (string * exp) list
let rec interp env e =
 match e with
  Var s -> List.assoc s env (* do the lookup *)
 Lam(s,e2) -> Closure(s,e2,env) (* store env! *)
 | Closure -> e (* closures are values *)
 | Apply(e1,e2) ->
   let v1 = interp env e1 in
    let v_2 = interp env e2 in
   match v1 with
     Closure(s,e3,env2) -> interp((s,v2)::env2) e3
     -> failwith "impossible"
```

Worth repeating

- A closure is a pair of code and environment
 - Implementing higher-order functions is not magic or run-time code generation
- An okay way to think about OCaml
 - Like thinking about OOP in terms of vtables
- Need not store whole environment of course
 - See Homework 3

What will we do

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 - Notes on concrete syntax (relates to OCaml)
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- Define substitution precisely
 - And revisit function equivalences
- Environments

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