CSE 599d - Quantum Computing
When Quantum Computers Fall Apart

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In this lecture we are going to begin discussing what happens to quantum computers when we move away from the closed system view of quantum theory and encounter the reality that real quantum system are open quantum systems. Open quantum systems interact with their environment and decohere. When we try to control real quantum systems, we aren't able to perfectly control them. When we try to perform projective measurements on a real quantum systems, we don't perform exactly perfect projective measurements. When we try to prepare real quantum systems into a particular state, we don't succeed in preparing this state with perfect certainty. All of these issues must be addressed if we are really going to (1) build a quantum computer, (2) accept that quantum computation is a valid model deserving of the moniker digital computer.

I.   QUANTUM NOISE

Suppose that our happy qubit $|\psi\rangle$ is sitting there minding its business, when the cold hard reality that is not alone in the universe shows its head. In particular another qubit, in the state $\sqrt{1-p}|0\rangle + \sqrt{p}|1\rangle$ comes along and interacts with our qubit by a controlled-NOT, controlled from the extra qubit. What will the effect of this evolution be on our qubit? Well we can calculate the Kraus operators:

$$
\begin{aligned}
A_0 &= \langle 0|_B C_X \left(\sqrt{1-p}|0\rangle_B + \sqrt{p}|1\rangle_B \right) = \sqrt{1-p}\,I \\
A_1 &= \langle 1|_B C_X \left(\sqrt{1-p}|0\rangle_B + \sqrt{p}|1\rangle_B \right) = \sqrt{p}\,X
\end{aligned}
\tag{1}
$$

The evolution of our initial state $\rho = |\psi\rangle\langle\psi|$ is then given by

$$
\rho \rightarrow A_0 \rho A_0 + A_1 \rho A_1 = (1-p)\rho + pX\rho X.
\tag{2}
$$

We can interpret this evolution as describing the procedure of doing nothing to our qubit with probability $1-p$ and with probability p applying the unitary operator X to our qubit. This is an example of quantum noise on our qubit. But notice that this quantum noise is very similar to what we might call classical noise in the computational basis. In the computational basis, this represents nothing more that flipping the bit.

Now suppose that we replace the controlled-NOT in this operation with a controlled-U operation. Then we can similarly calculate that

$$
\begin{aligned}
A_0 &= \langle 0|_B C_X \left(\sqrt{1-p}|0\rangle_B + \sqrt{p}|1\rangle_B \right) = \sqrt{1-p}\,I \\
A_1 &= \langle 1|_B C_X \left(\sqrt{1-p}|0\rangle_B + \sqrt{p}|1\rangle_B \right) = \sqrt{p}\,U
\end{aligned}
\tag{3}
$$

The evolution of this superoperator can be interpreted as doing nothing with probability $1-p$ and applying U with probability p. Now if U is the Z gate, then, this evolution does something kind of strange to our qubit, when expressed in the computational basis. In particular the Z gate does not change the amplitude of a state in the computational basis: $\alpha|0\rangle + \beta|1\rangle \rightarrow \alpha|0\rangle - \beta|1\rangle$. Thus if we are talking about measurements in the computational basis the effect of this "noise" does not change the probabilities of the two outcomes. But if we examine the density matrix, then something has happened to our system:

$$
\begin{bmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{bmatrix} \rightarrow (1-p) \begin{bmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{bmatrix} + p \begin{bmatrix} |\alpha|^2 & -\alpha\beta^* \\ -\alpha^*\beta & |\beta|^2 \end{bmatrix} = \begin{bmatrix} |\alpha|^2 & (1-2p)\alpha\beta^* \\ (1-2p)\alpha^*\beta & |\beta|^2 \end{bmatrix}
\tag{4}
$$

Thus we see that the off diagonal elements of the the density matrix have decayed. If, for example $p = \frac{1}{2}$, then the density matrix has gone from a coherent mixture to an incoherent mixture. This is a particulary nasty form of quantum noise, because if, say $|0\rangle$ and $|1\rangle$ are energy eigenstates, then the energy of the qubit has not changed, but the system has lost some of it's coherent properties.

Now what is bad about the quantum noise we have described above? Well the real problem is that it is not reversible. Suppose that the interactions we describe above happen, but then the extra qubit with with our qubit

interacted becomes inaccessible to us. Well then can we restore our qubit back to it's original state? The answer is no!

Why? Well one way to see this to examine the action of the noise processes which can occur to a qubit on the Bloch ball. A process like our first noise process, produces the evolution

$$\frac{1}{2}(I + n_x X + n_y Y + n_z Z) \rightarrow \frac{1-p}{2}(I + n_x X + n_y Y + n_z Z) + \frac{p}{2}(I + n_x X - n_y Y - n_z Z)$$

$$= \frac{1}{2}(I + n_x X - (1 - 2p)(n_y Y + n_z Z)). \tag{5}$$

Thus we see that the states in the Bloch ball are shrunk around the Y and Z directions (into a Bloch ellipsoid.) If we are to reverse this shrinking, there must be a process which reverses this shrinking. But no such valid operator sum representation evolution can produce this. Why? Because it would need to take states with $|\vec{n}| < 1$ to states with $|\vec{n}| = 1$. But superoperators are linear, $\$[aM + bN] = a\$[M] + b\$[N]$. So if the there exists a superoperator which performs this for our above operation, it must take $\frac{1}{2}(I \pm (1-2p)Y) \rightarrow \frac{1}{2}(I \pm Y)$. This implies $\frac{1}{2}(\$[I] + (1-2p)\$[Y]) = \frac{1}{2}(I + Y)$ and $\frac{1}{2}(\$[I] - (1-2p)\$[Y]) = \frac{1}{2}(I - Y)$, or $(1-2p)\$[Y] = Y$ and $\$[I] = I$. So applying this to superoperator to $\frac{1}{2}(I + Y)$ we obtain $\frac{1}{2}(I + \frac{1}{1-2p}Y)$ which is not a valid density matrix from $p > 0$. Thus, since superoperators map density matrices to density matrices, we have obtained a contradiction, and there must exist no such superoperator which performs this map. Thus we see that the above process we have described is irreversible, given that we throw away the system with which our system has interact to produce the original superoperator.

Let's describe a few of the more important mechanism which we encounter in studying quantum systems. In particular we will focus on the action of these quantum noise processes on a single qubit.

A. Depolarizing Channel

In a depolarizing channel, with probability $1-p$ the qubit is left untouched and with probability p the qubit is fully mixed. Fully mixed? This means that $\frac{1}{2}(I + \vec{n} \cdot \sigma) \rightarrow \frac{1}{2}I$. How can we achieve this process? One way is to note that

$$X\rho X + Y\rho Y + Z\rho Z = \frac{3}{2}I \tag{6}$$

To seen this note that $XXX = X$, $XYX = -X$, and $XZX = -Z$ so that if $\rho = \frac{1}{2}(I + n_x X + n_y Y + n_z Z)$, then $X\rho X = \frac{1}{2}(I + n_x X - n_y Y - n_z Z)$. Similarly $Y\rho Y = \frac{1}{2}(I - n_x X + n_y Y - n_z Z)$ and $Z\rho Z = \frac{1}{2}(I - n_x X - n_y Y + n_z Z)$. Putting these together yeilds the expression. Thus we see that we can produce the depolarizing channel if we have the four Kraus operators

$$A_0 = \sqrt{1-p}I, \quad A_1 = \sqrt{\frac{p}{3}}X \quad A_2 = \sqrt{\frac{p}{3}}Y \quad A_3 = \sqrt{\frac{p}{3}}Z \tag{7}$$

Then the evolution of ρ will be

$$\rho \rightarrow (1-p)\rho + p\frac{1}{2}I \tag{8}$$

What does this do on the Bloch ball? Well it simply shrinks the ball along all directs equally:

$$\vec{n} \rightarrow (1-p)\vec{n} \tag{9}$$

B. Phase Damping Channel

Suppose that our qubit is sitting there, minding its business. The environment, however is always out there and always planning devious business for our qubit. Suppose that the environment is in a state $|0\rangle$ but every once in a while, it scatters off of the qubit. In particular this scattering may depend on the state of the qubit, i.e. if the qubit is in the $|0\rangle$ state, then there may be an amplitude that the environment scatters into the state $|1\rangle$ and if the qubit is in the $|1\rangle$ state, there may be a different amplitude that the environment scatters into the state $|2\rangle$. Consider, for example, the situation where the magnitude of the evolution is the same:

$$|0\rangle \otimes |0\rangle \rightarrow |0\rangle \otimes (\sqrt{1-p}|0\rangle + \sqrt{p}|1\rangle)$$
$$|1\rangle \otimes |0\rangle \rightarrow |1\rangle \otimes (\sqrt{1-p}|0\rangle + \sqrt{p}|2\rangle) \tag{10}$$

The Kraus operators for this process can be calculated, assuming the environment starts in the state $|0\rangle$, and is given by

$$A_0 = \begin{bmatrix} \sqrt{1-p} & 0 \\ 0 & \sqrt{1-p} \end{bmatrix} \quad A_1 = \begin{bmatrix} \sqrt{p} & 0 \\ 0 & 0 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{p} \end{bmatrix} \tag{11}$$

What does this evolution do to an arbitrary input density matrix? It is easy to see that it only effects the off diagonal matrix elements:

$$\begin{bmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{bmatrix} \rightarrow (1-p)\begin{bmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{bmatrix} + p\begin{bmatrix} |\alpha|^2 & 0 \\ 0 & |\beta|^2 \end{bmatrix} = \begin{bmatrix} |\alpha|^2 & (1-p)\alpha\beta^* \\ (1-p)\alpha^*\beta & |\beta|^2 \end{bmatrix} \tag{12}$$

But notice that this is the same as the evolution we described by coupling a controlled-Z to our system. But that evolution had different Kraus operators. That's kind of interesting. In fact these two evolutions, which are call phase damping, represent (up to that scaling) the same superoperator. There is a degree of freedom in our superoperators which we haven't discussed, and we will now remedy that situation

1. Unitary Freedom of the Operator Sum Representation

To understand the freedom in the operator sum representation Kraus operators, it is useful to first understand a very cool way to fully characterize a superoperator. Suppose we have some superoperator $ with Kraus operator A_i. Now one way to characterize this superoperator is to check how it operators on different states and, given enough of these states, one can regain all of the information describing the superoperator. This is known as process tomography, and we won't discuss it here. But there is a much cooler way to specify a superoperator.

Let our system have a Hilbert space $\mathcal{H}_A$. Attach an ancilla system of the same dimension as our Hilbert space to $\mathcal{H}_A$ so our full Hilbert space is $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. Now define the (unnormalized) state on this joint system

$$|\psi\rangle_{AB} = \sum_{i=1}^{d} |i\rangle_A \otimes |i\rangle_B \tag{13}$$

where $d = \dim\mathcal{H}_A$. This is the maximally entangled states (if we had properly normalized it) shared between $\mathcal{H}_A$ and $\mathcal{H}_B$. Now the cool thing about $|\psi\rangle$ is that it allows us to specify states on subsystem A using states on subsystem B. In particular note that is we let $|\phi\rangle_B = \sum_{i=1}^{d} a_i^*|i\rangle_B$, then

$$\langle\phi|_B|\psi\rangle_{AB} = \sum_{i=1}^{d} a_i|i\rangle_A = |\tilde{\phi}\rangle_B. \tag{14}$$

Evidently, $|\psi\rangle_{AB}$ can serve as a map from states on B to states on A. What consequence does this have for superoperators? Well define the operator

$$\tau = (\$ \otimes I)[|\psi\rangle_{AB}\langle\psi|_{AB}] \tag{15}$$

Now what is cool is that τ gives us all of the information about $ that we need to reconstruct $! By looking at how a superoperator acts on half of a maximally entangled quantum state, we can completely characterize the superoperator.

How do we see this? Well again defining $|\phi\rangle_B = \sum_{i=1}^{d} a_i^*|i\rangle_B$, we can calculate $\langle\phi|_B\tau|\phi\rangle_B$

$$\langle\phi|_B\tau|\phi\rangle_B = \langle\phi|_B(\$ \otimes I)[|\psi\rangle_{AB}\langle\psi|_{AB}]|\phi\rangle_B = (\$)[\langle\phi|_B|\psi\rangle_{AB}\langle\psi|_{AB}|\phi\rangle_B] = \$[|\tilde{\phi}\rangle_A\langle\tilde{\phi}|_A] \tag{16}$$

Suppose that we decompose τ as $\tau = \sum_k |v_k\rangle_{AB}\langle v_k|_{AB}$. Then define map $A_k(|\phi\rangle_A) = \langle\tilde{\phi}|_B|v_k\rangle_{AB}$. This map is certainly linear, so we can just think about it as a linear operator. But then we have

$$\sum_k A_k|\phi\rangle_A\langle\phi|_A A_k^\dagger = \sum_k \langle\tilde{\phi}|_B|v_k\rangle_{AB}\langle v_k|_{AB}|\tilde{\phi}\rangle_B = \langle\tilde{\phi}|_B\tau|\tilde{\phi}\rangle_B = \$[|\phi_A\rangle\langle\phi|_A] \tag{17}$$

Thus we have seen that by seeing how $ acts on half of a maximally entangled state, we can calculate τ, and using τ we can then find the operator sum representation of $.

Okay now that we've got that cool result out of the way, we can discuss the freedom in the operator sum representation. Suppose that we have two superoperators with Kraus operator $\{A_k\}$ and $\{B_k\}$. Assume that they have the

same number of operators and if they do not, then just pad on of the sets with zero operators. Then we will show that if these represent the same operator then these represent the same superoperator iff there exists a unitary matrix with entries u_{mn} such that

$$A_k = \sum_l u_{kl} B_l \tag{18}$$

Again attach an ancilla space B with the same dimension as A. Then define the states resulting from acting with A_k or B_l on one half of (not properly normalized) maximally entangled state between A and B.

$$
\begin{aligned}
|a_k\rangle &= \sum_{i=1}^{d} (A_k |i\rangle_A) \otimes |i\rangle_B \\
|b_l\rangle &= \sum_{i=1}^{d} (B_l |i\rangle_A) \otimes |i\rangle_B
\end{aligned}
\tag{19}
$$

Then it must be that the operator which was τ above, representing the result of acting with $I \otimes \$$ on the maximally entangled state, are, for these two superoperators, $\tau = \sum_k |a_k\rangle\langle a_k|$ and $\sigma_l = \sum_l |b_l\rangle\langle b_l|$. But these are (up to that normalization) valid density matrices. The only way these can be the same density matrices and hence represent the same superoperator is, from a few lectures back, if

$$|a_k\rangle = \sum_l u_{kl} |b_l\rangle \tag{20}$$

for some unitary matrix with entries u_{kl}. Now for arbitrary states $|\psi\rangle$, we find that

$$A_k |\psi\rangle_A = \langle\tilde\psi|_B |a_k\rangle = \sum_l \langle\tilde\psi|_B u_{kl} |b_l\rangle = \sum_l u_{kl} B_l |\psi\rangle \tag{21}$$

Since this must hold for arbitrary $|\psi\rangle$ this implies that

$$A_k = \sum_l u_{kl} B_l \tag{22}$$

as desired.

Conversely, suppose that $A_k = \sum_l u_{kl} B_l$. Then $\sum_k A_k \rho A_k^\dagger = \sum_k \sum_l u_{kl} B_l \rho \sum_m u_{km}^* B_m$. But $\sum_k u_{kl} u_{km}^* = \delta_{lm}$, so this implies that $\sum_k A_k \rho A_k^\dagger = \sum_l B_l \rho B_l^\dagger$.

Thus we have shown that two superoperators are the same iff the sets of Kraus operators for these superoperators can be related to each other by a unitary transform over the different operators (not on the operators themselves.)

Let's see this for the two channels where we first noted this. In the first superoperator the Krauss operators were

$$A_0 = \sqrt{1 - \frac{p}{2}} I \quad A_1 = \sqrt{\frac{p}{2}} Z \tag{23}$$

and in the second case the Kraus operators were

$$B_0 = \begin{bmatrix} \sqrt{1-p} & 0 \\ 0 & \sqrt{1-p} \end{bmatrix} \quad B_1 = \begin{bmatrix} \sqrt{p} & 0 \\ 0 & 0 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{p} \end{bmatrix} \tag{24}$$

(where we have rescaled the first to make this correspondence correct.) We can rewrite these second Kraus operators as

$$B_0 = \sqrt{1-p} I \quad B_1 = \frac{\sqrt{p}}{2}(I + Z) \quad B_2 = \frac{\sqrt{p}}{2}(I - Z) \tag{25}$$

Then it the question becomes whether we can find a unitary which transforms between these two sets o matrices. Indeed we can, and it is given by

$$U = \begin{bmatrix} \sqrt{\frac{2(1-p)}{2-p}} & \sqrt{\frac{p}{2(2-p)}} & \sqrt{\frac{p}{2(2-p)}} \\ 0 & \frac{1}{\sqrt 2} & -\frac{1}{\sqrt 2} \\ -\sqrt{\frac{p}{2-p}} & \sqrt{\frac{1-p}{2-p}} & \sqrt{\frac{1-p}{2-p}} \end{bmatrix} \tag{26}$$

How did I find this? Well I looked at the coefficients in from of the I and Z separately and derived equations from these conditions. Of course the answer I got is not unique: the final row can have an arbitrary phase, for example.

C.   Amplitude Damping Channel

A final, very interesting channel is the amplitude damping channel. This channel is used to describe, say the decay of an a two level system. Suppose that our environment is initially in the state $|0\rangle_E$ representing the vacuum. Then if the system is in its lower energy state, which we take to be $|0\rangle$, then nothing happens. However if the system is in its higher energy state, which we take to be the $|1\rangle$ state, then there is an amplitude for the system to relax to the ground state and emit a photon. The unitary description of this process is

$$
\begin{aligned}
|0\rangle \otimes |0\rangle_E &\rightarrow |0\rangle \otimes |0\rangle_E \\
|1\rangle \otimes |0\rangle_E &\rightarrow \sqrt{1-p}|1\rangle \otimes |0\rangle_E + \sqrt{p}|0\rangle \otimes |1\rangle_E
\end{aligned}
\tag{27}
$$

Calculating the Kraus operators we find that they are given by

$$
A_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{bmatrix} \quad A_1 = \begin{bmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{bmatrix}
\tag{28}
$$

Notice that A_1 is not a hermitian operator (as in our previous Kraus operators.) How does a density matrix evolve under this evolution? Well we can calculate that

$$
\begin{bmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{bmatrix} \rightarrow \begin{bmatrix} |\alpha|^2 & \alpha\beta^*\sqrt{1-p} \\ \alpha^*\beta\sqrt{1-p} & |\beta|^2(1-p) \end{bmatrix} + \begin{bmatrix} p|\beta|^2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} |\alpha|^2(1-p)+|\beta|^2 p & \sqrt{1-p}\alpha\beta^* \\ \sqrt{1-p}\alpha^*\beta & |\beta|^2(1-p) \end{bmatrix}
\tag{29}
$$

Thus we see that there is population transfer between the $|1\rangle$ and $|0\rangle$ state, along with an damping of the off diagonal elements of the density matrix. If p represents the amplitude for this to occur per unit time step, for a time t, then the population of the higher energy level state decays like $(1-p\Delta t)^{\frac{t}{\Delta t}} \approx \exp(-pt)$. So this is just the exponential decay of a higher energy state to a lower energy state.

Notice also that the amplitude damping channel does not send $\frac{1}{2}I$ to $\frac{1}{2}I$. Quantum noise which preserve the maximally mixed state (like $\frac{1}{2}I$ for a qubit) are called unital. Thus amplitude damping is not a unital superoperator.

D.   Quantum Noise

We've seen some possible quantum noise processes in this lecture. The question we will worry about next is how we might possibly overcome this quantum noise. One particular worry is that the noise is specified by Kraus operator $\{A_k\}$ and these form a continuous set of operators. Of course, if we really think about classical noise, then it will also form a continuous set. We aren't worried there because we can think about noise in the classical case as deterministic procedures occurring with probabilities. Are their equivalent notions for quantum theory? Stay tuned and you will find out!