Case Study 1: Estimating Click Probabilities

Perceptron Algorithm
Kernels (continued)

Machine Learning/Statistics for Big Data
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Online Learning Problem

- At each time step t:
  - Observe features of data point:
    - Note: many assumptions are possible, e.g., data is iid, data is adversarially chosen… details beyond scope of course

$x^{(t)}$

  - Make a prediction:
    - Note: many models are possible, we focus on linear models
    - *For simplicity, use vector notation*

predict $\hat{y}^{(t)} \approx y$

$x^{(t)} = \begin{pmatrix} x^{(t)} \\ 1 \end{pmatrix}$

$w_0 + \sum_i w_i x_i^{(t)} > 0$? $\Rightarrow$ $w^{(t)} \cdot x^{(t)} > 0$

$\sum_{i=0}^{d} w_i x_i^{(t)} \rightarrow x_0^{(t)} = 1$

  - Observe true label:
    - Note: other observation models are possible, e.g., we don't observe the label directly, but only a noisy version... Details beyond scope of course

observe $y^{(t)} \rightarrow$ clicked or not clicked

... what is Δ ??

  - Update model:

$w^{(t+1)} \leftarrow w^{(t)} + \Delta^{(t)}$ something

The Perceptron Algorithm [Rosenblatt '58, '62]

- Classification setting: y in {-1,+1}
- Linear model
  - Prediction: $\hat{y} = \mathrm{sign}(w \cdot x)$
- Training:
  - Initialize weight vector: $w^{(0)} = 0$
  - At each time step:
    - Observe features: $x^{(t)} \leftarrow$ user, page, ad features
    - Make prediction: $\hat{y} = \mathrm{sign}(w^{(t)} \cdot x^{(t)})$
    - Observe true class:

      $y^{(t)} \leftarrow$ true label
    - Update model:
      - If prediction is not equal to truth

        if make a mistake

        if $\hat{y} \neq y^{(t)}$

        $w^{(t+1)} \leftarrow w^{(t)} + y^{(t)} x^{(t)}$

        else: $w^{(t+1)} \leftarrow w^{(t)}$

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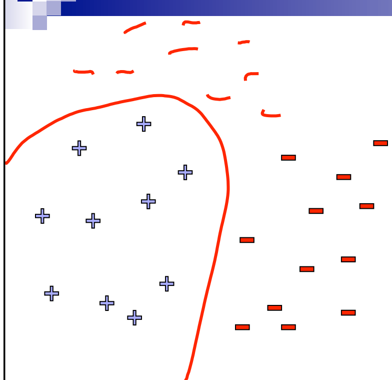
What if the data is not linearly separable?



Use features of features of features of features….

$$\Phi(\mathbf{x}) : R^m \mapsto F$$

$$\phi(x) = \begin{pmatrix} x \\ x^2 \\ x^3 \\ x^4 \\ e^{-x} \\ e^{-\sin \log \cos x} \\ \vdots \end{pmatrix}$$

Feature space can get really large really quickly!

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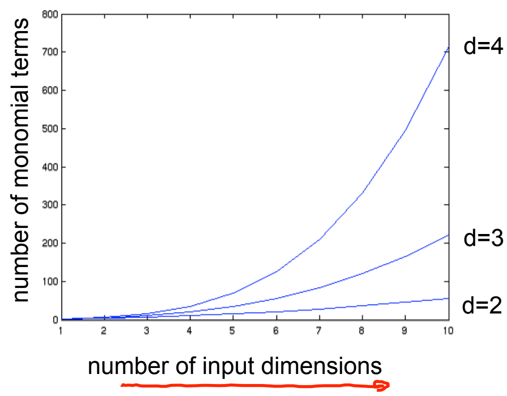
Higher order polynomials

$$\text{num. terms} = \binom{d+m-1}{d} = \frac{(d+m-1)!}{d!(m-1)!}$$

dim of ϕ



m – input features
d – degree of polynomial

grows fast!
d = 6, m = 100
about 1.6 billion terms

Perceptron Revisited

- Given weight vector $w^{(t)}$, predict point $\mathbf{x}$ by:

$$\hat{y} = \text{Sign}(w^{(t)} \cdot x)$$

- Mistake at time t: $w^{(t+1)} = w^{(t)} + y^{(t)} x^{(t)}$
- Thus, write weight vector in terms of mistaken data points only:
  - Let $M^{(t)}$ be time steps up to t when mistakes were made:

$$w^{(t)} = \sum_{i \in M(t)} y^{(i)} x^{(i)}$$

- Prediction rule now:

$$\text{Sign}(w^{(t)} \cdot x) = \text{Sign}\left(\left(\sum_{i \in M(t)} y^{(i)} x^{(i)}\right) \cdot x\right) = \text{Sign}\left(\sum_{i \in M(t)} y^{(i)} x^{(i)} \cdot x\right)$$

- When using high dimensional features: ← list of mistakes ever made

$$\text{Sign}\left(\sum_{i \in M(t)} y^{(i)} \underbrace{\phi(x^{(i)}) \cdot \phi(x)}_{\text{dot product between } x \text{ and mistake } i}\right)$$

Dot-product of polynomials

$u = (u_1, u_2)$
$v = (v_1, v_2)$

$\Phi(\mathbf{u}) \cdot \Phi(\mathbf{v}) =$ polynomials of degree exactly d

$d=1$ $\quad \phi(u)\cdot\phi(v) = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = u_1 v_1 + u_2 v_2 = u \cdot v$

$d=2$ $\quad \phi(u)\cdot\phi(v) = \begin{pmatrix} u_1^2 \\ u_1 u_2 \\ u_2 u_1 \\ u_2^2 \end{pmatrix} \cdot \begin{pmatrix} v_1^2 \\ v_1 v_2 \\ v_2 v_1 \\ v_2^2 \end{pmatrix} = u_1^2 v_1^2 + 2 u_1 u_2 v_1 v_2 + u_2^2 v_2^2 = (u_1 v_1 + u_2 v_2)^2 = (u \cdot v)^2$

proof by one step of induction

if $\phi(\cdot)$ is poly of degree exactly d,

$\phi(u) \cdot \phi(v) = (u \cdot v)^d$

compute in time of basically $u \cdot v$

Finally the Kernel Trick!!! (Kernelized Perceptron

- Every time you make a mistake, remember $(x^{(t)}, y^{(t)})$
- Kernelized Perceptron prediction for **x**:

$$\mathrm{sign}(\mathbf{w}^{(t)} \cdot \phi(\mathbf{x})) = \sum_{i \in M^{(t)}} \phi(\mathbf{x}^{(i)}) \cdot \phi(\mathbf{x})$$
$$= \sum_{i \in M^{(t)}} k(\mathbf{x}^{(i)}, \mathbf{x})$$

Polynomial kernels

- All monomials of degree d in O(d) operations:

$$\Phi(\mathbf{u})\cdot\Phi(\mathbf{v}) = (\mathbf{u}\cdot\mathbf{v})^d = \text{polynomials of degree exactly d}$$

- How about all monomials of degree up to d?
  - Solution 0:
  - Better solution:

Common kernels

- Polynomials of degree exactly d

$$K(\mathbf{u},\mathbf{v}) = (\mathbf{u}\cdot\mathbf{v})^d$$

- Polynomials of degree up to d

$$K(\mathbf{u},\mathbf{v}) = (\mathbf{u}\cdot\mathbf{v}+1)^d$$

- Gaussian (squared exponential) kernel

$$K(\mathbf{u},\mathbf{v}) = \exp\left(-\frac{||\mathbf{u}-\mathbf{v}||}{2\sigma^2}\right)$$

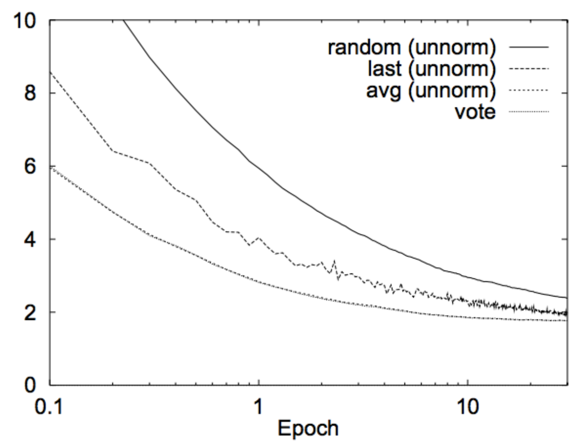
- Sigmoid

$$K(\mathbf{u},\mathbf{v}) = \tanh(\eta\mathbf{u}\cdot\mathbf{v}+\nu)$$

Fundamental Practical Problem for All Online Learning Methods: **Which weight vector to report?**

- Suppose you run online learning method and want to sell your learned weight vector… Which one do you sell???
- Last one?
-
-
-

Choice can make a huge difference!!



[Freund & Schapire '99]

What you need to know

- Notion of online learning
- Perceptron algorithm
- Mistake bounds and proofs
- The kernel trick
- Kernelized Perceptron
- Derive polynomial kernel
- Common kernels
- In online learning, report averaged weights at the end

Case Study 1: Estimating Click Probabilities

Stochastic Gradient Descent

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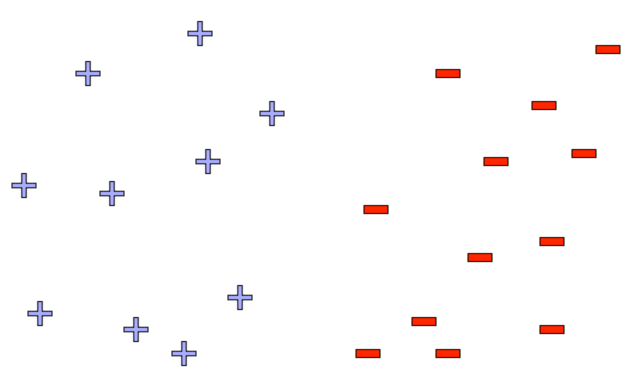
What is the Perceptron Doing???

- When we discussed logistic regression:
  - Started from maximizing conditional log-likelihood
- When we discussed the Perceptron:
  - Started from description of an algorithm
- What is the Perceptron optimizing????

Perceptron Prediction: Margin of Confidence



Hinge Loss

- Perceptron prediction:
- Makes a mistake when:
- Hinge loss (same as maximizing the margin used by SVMs)

Minimizing hinge loss in Batch Setting

- Given a dataset:
- Minimize average hinge loss:
- How do we compute the gradient?

Subgradients of Convex Functions

- Gradients lower bound convex functions:
- Gradients are unique at **x** if function differentiable at **x**
- Subgradients: Generalize gradients to non-differentiable points:
  - Any plane that lower bounds function:

Subgradient of Hinge

- Hinge loss:
- Subgradient of hinge loss:
  - If $y^{(t)}(w.\mathbf{x}^{(t)}) > 0$:
  - If $y^{(t)}(w.\mathbf{x}^{(t)}) < 0$:
  - If $y^{(t)}(w.\mathbf{x}^{(t)}) = 0$:
  - In one line:

Announcements

- No recitation this week
- Comments on readings:
  - Material in readings are superset of what you need
  - Read foundations, e.g., from Kevin Murphy's book, before class
  - Fill in details after class
- Homework out today
  - Start early, start early, start early, start early, start early, start early, start early, start early, start early, start early, start early, start early, start early…
  - Warm-up part of programming due on 1/22
  - Full homework due on 1/29, beginning of class

Subgradient Descent for Hinge Minimization

- Given data:
- Want to minimize:
- Subgradient descent works the same as gradient descent:
  - But if there are multiple subgradients at a point, just pick (any) one:

Perceptron Revisited

- Perceptron update:

$$\mathbf{w}^{(t+1)} \leftarrow \mathbf{w}^{(t)} + \mathbb{1}\left[y^{(t)}(\mathbf{w}^{(t)} \cdot \mathbf{x}^{(t)}) \leq 0\right] y^{(t)}\mathbf{x}^{(t)}$$

- Batch hinge minimization update:

$$\mathbf{w}^{(t+1)} \leftarrow \mathbf{w}^{(t)} + \eta \frac{1}{N} \sum_{i=1}^{N} \left\{ \mathbb{1}\left[y^{(i)}(\mathbf{w}^{(t)} \cdot \mathbf{x}^{(i)}) \leq 0\right] y^{(i)}\mathbf{x}^{(i)} \right\}$$

- Difference?

Learning Problems as Expectations

- Minimizing loss in training data:
  - Given dataset:
    - Sampled iid from some distribution p(**x**) on features:
  - Loss function, e.g., hinge loss, logistic loss,…
  - We often minimize loss in training data:

$$\ell_{\mathcal{D}}(\mathbf{w}) = \frac{1}{N} \sum_{i=1}^{N} \ell(\mathbf{w}, \mathbf{x}^{(i)})$$

- However, we should really minimize expected loss on all data:

$$\ell(\mathbf{w}) = E_{\mathbf{x}}\left[\ell(\mathbf{w}, \mathbf{x})\right] = \int p(\mathbf{x})\ell(\mathbf{w}, \mathbf{x})d\mathbf{x}$$

- So, we are approximating the integral by the average on the training data

Gradient descent in Terms of Expectations

- "True" objective function:

$$\ell(\mathbf{w}) = E_{\mathbf{x}}\left[\ell(\mathbf{w},\mathbf{x})\right] = \int p(\mathbf{x})\ell(\mathbf{w},\mathbf{x})d\mathbf{x}$$

- Taking the gradient:
- "True" gradient descent rule:
- How do we estimate expected gradient?

SGD: Stochastic Gradient Descent (or Ascent)

- "True" gradient: $\nabla\ell(\mathbf{w}) = E_{\mathbf{x}}\left[\nabla\ell(\mathbf{w},\mathbf{x})\right]$
- Sample based approximation:
- What if we estimate gradient with just one sample???
  - Unbiased estimate of gradient
  - Very noisy!
  - Called stochastic gradient descent
    - Among many other names
  - VERY useful in practice!!!

Perceptron & Stochastic Gradient descent

- Perceptron update:

$$\mathbf{w}^{(t+1)} \leftarrow \mathbf{w}^{(t)} + \mathbb{1}\left[y^{(t)}(\mathbf{w}^{(t)} \cdot \mathbf{x}^{(t)}) \leq 0\right] y^{(t)}\mathbf{x}^{(t)}$$

- Batch hinge minimization update:

$$\mathbf{w}^{(t+1)} \leftarrow \mathbf{w}^{(t)} + \eta \frac{1}{N} \sum_{i=1}^{N} \left\{\mathbb{1}\left[y^{(i)}(\mathbf{w}^{(t)} \cdot \mathbf{x}^{(i)}) \leq 0\right] y^{(i)}\mathbf{x}^{(i)}\right\}$$

Stochastic Gradient Descent: general case

- Given a stochastic function of parameters:
  - Want to find minimum
- Start from $\mathbf{w}^{(0)}$
- Repeat until convergence:
  - Get a sample data point $\mathbf{x}^{(t)}$
  - Update parameters:
- Works on the online learning setting!
- Complexity of gradient computation is constant in number of examples!
- In general, step size changes with iterations

Stochastic Gradient Ascent for Logistic Regression

- Logistic loss as a stochastic function:

$$E_{\mathbf{x}}\left[\ell(\mathbf{w},\mathbf{x})\right] = E_{\mathbf{x}}\left[\ln P(y|\mathbf{x},\mathbf{w}) - \lambda||\mathbf{w}||_2^2\right]$$

- Batch gradient ascent updates:

$$w_i^{(t+1)} \leftarrow w_i^{(t)} + \eta\left\{-\lambda w_i^{(t)} + \frac{1}{N}\sum_{j=1}^{N} x_i^{(j)}[y^{(j)} - P(Y=1|\mathbf{x}^{(j)},\mathbf{w}^{(t)})]\right\}$$

- Stochastic gradient ascent updates:
  - Online setting:

$$w_i^{(t+1)} \leftarrow w_i^{(t)} + \eta_t\left\{-\lambda w_i^{(t)} + x_i^{(t)}[y^{(t)} - P(Y=1|\mathbf{x}^{(t)},\mathbf{w}^{(t)})]\right\}$$

Convergence rate of SGD

- **Theorem**:
  - (see Nemirovski et al '09 from readings)
  - Let *f* be a strongly convex stochastic function
  - Assume gradient of *f* is Lipschitz continuous and bounded
  - Then, for step sizes:
  - The expected loss decreases as O(1/t):

Convergence rates for gradient descent/ascent versus SGD

- Number of Iterations to get to accuracy

$$\ell(\mathbf{w}^*) - \ell(\mathbf{w}) \leq \epsilon$$

- Gradient descent:
  - If func is strongly convex: $O(\ln(1/\epsilon))$ iterations
- Stochastic gradient descent:
  - If func is strongly convex: $O(1/\epsilon)$ iterations
- Seems exponentially worse, but much more subtle:
  - Total running time, e.g., for logistic regression:
    - Gradient descent:
    - SGD:
    - SGD can win when we have a lot of data
  - And, when analyzing true error, situation even more subtle… expected running time about the same, see readings

What you need to know

- Perceptron is optimizing hinge loss
- Subgradients and hinge loss
- (Sub)gradient decent for hinge objective
- Objective functions in ML as expectations
- Gradient estimation, rather than objective estimation
- Stochastic gradient descent -> estimate gradient from single training example
  - Mini-batches possible and useful
- Stochastic gradient ascent for logistic regression
- Analysis of stochastic gradient descent
  - Decreasing step size fundamental here
- Comparing analysis of stochastic gradient descent with gradient descent