Intermediate motor learning as decreasing active (dynamical) degrees of freedom

Suvobrata Mitra a*, Polemnia G. Amazeen b, M.T. Turvey a

a Center for the Ecological Study of Perception and Action, Department of Psychology, U-20 University of Connecticut, 406 Babidge Road, Storrs, CT 06269, USA

b Faculty of Human Movement Sciences, Vrije Universiteit, Amsterdam, The Netherlands

Abstract

A classical view is that motor learning has distinguishable early, intermediate, and late phases. A recent view is that motor learning is the acquisition of an abstract equation of motion that specifies the time evolution of a pattern of coordination. The pattern is expressed by a collective variable that enslaves or orders component subsystems that, in turn, act on and generate the collective variable. In these latter terms, early learning resolves the collective variable and its motion equation, intermediate learning stabilizes and standardizes the subsystems or active degrees of freedom (DFs) producing the collective variable’s dynamics. The preceding ideas, and the phase–space reconstruction methods required to determine active DFs, are developed in tutorial fashion in the context of an experimental investigation of learning a bimanual rhythmic coordination. Results show that intermediate learning reduces the dimensionality of the learned coordination’s dynamics and renders those dynamics more deterministic. The tutorial development relates the preceding concepts, results and methods of analyses to (a) the contrast between Poincaréan and Newtonian dynamics, (b) contemporary interpretations of random processes, (c) definitions of DFs in respect to Bernstein’s problem, (d) the potential contribution of chaos to the adaptability of a learned coordination, and (e) possible links between active (dynamical) DFs and the control variables $r$, $c$, and $\mu$ identified by the $\lambda$ hypothesis. © 1998 Elsevier Science B.V.

* Corresponding author. Tel.: +1 860 486 0998; e-mail: joy@indra.psy.uconn.edu.
1. Introduction: What is learned?

When a new coordination is learned, what in fact has been acquired? Some years ago Fowler and Turvey (1978) suggested that in motor learning the person discovers how to fashion him or herself into that particular kind of dynamical system that is consonant with the essential features of the skill. To learn a motor skill is to acquire a functionally specific, special-purpose organization of the relevant variables of the movement system, the task, and the environment (see also Newell, 1986; Saltzman and Kelso, 1987). In the present article, the special-purpose organization is referred to as a coordination dynamic. It can be expressed as an equation of motion of the form

$$\dot{\xi} = L[\xi, c; F].$$  \hspace{1cm} (1)

In this first-order, autonomous, differential equation, $\xi$ is a collective variable capturing the specific, spatial and temporal details of the coordination. Because it captures, in unitary fashion, the pattern of coordination, $\xi$ is unlikely to be a commonplace variable of mechanics or biomechanics. Rather, it is likely to be an abstract variable specific to the coordination. The first time derivative of $\xi$, identified by the overdot, expresses the coordination pattern's evolution or change in time. This temporal variation is according to a rule or (better) a local dynamical law $L$, applied to the current value of the coordination $\xi(t)$ and the current value of one or more control parameters, $c$. During the execution of the coordination pattern, $c$ is constant (and across a range of values of $c$ the coordination pattern will remain constant, before giving way, at a critical $c$ value, to a new coordination pattern). The remaining term $F$ refers to random influences, chance events, arising from the multiple degrees of freedom (DFs) – the very many component subsystems – not absorbed by, or organized through, the collective variable $\xi$. There are, therefore, two aspects of Eq. (1): A deterministic aspect in which the time-evolution of the coordination is specified uniquely by the values of $\xi(t)$ and $c$, and a stochastic aspect $F$ that perturbs the systematically changing coordination. Roughly speaking, the process of learning a
new coordination characterized by a particular variant of Eq. (1) entails discovering the deterministic part (meaning $\xi$, $c$, and $L$) and reducing the stochastic part ($F$). Because learning never takes place in the absence of already available motor skills, discovering the deterministic part means finding how to change existing coordination dynamics to accommodate new requirements.

1.1. Single state spaces under high-level, abstract task descriptions

Answering the question of what is learned in the manner of Eq. (1) is due to Schöner and Kelso (1988a, b) (see also Kelso, 1995; Schöner, 1989; Schöner et al., 1992; Zanone and Kelso, 1992, 1994). At first, Eq. (1) may seem an odd way to characterize the thing learned. There is no mention of muscular forces and no mention of perceptual processes (in either prospective or retrospective roles). There is no referent for the biomechanical DFs of the moving limb segments, neither in respect to their identity nor in respect to their inertia. Yet it is apparent, to many past and contemporary commentators on skill learning, that a characterization of the consequences of motor learning cannot be addressed in anything less than a task-dependent, abstract form, devoid of specifics (e.g., Bernstein, 1967, 1996; Fitts, 1964; Humphrey, 1933; Lashley, 1951; Shaw and Alley, 1985; Schmidt, 1975). Greene (1975a, b, 1978) assumes that, as a general principle, both the entire family of circumstances in which a skill is performed and the entire family of movement patterns by which it is performed, comprise single state spaces under some high-level task description. Accordingly, every coordination is potentially a single function that maps one total space into the other (see Schmidt and Turvey, 1992). Greene’s arguments are directed at the notion of coordination as it applies to families of action variants and not just at the notion of coordination as it applies to a single action. For task-oriented motor-skill learners, the desideratum is “... somehow to pack all the relevant information about possibilities and variants into something that looks and behaves, at the high level, like a function from one set to another (Greene, 1978, pp. 1–11).” Ideally, this is precisely what a coordination dynamic provides. If motor learning endows a biological movement system with a behavioral potential of the form expressed by Eq. (1), then the system is in possession of a lawful way of producing a wide variety of systematically related, functionally specific patterns over different, relatively independent perception-action subsystems.
1.2. The geometry of dynamics and the challenge of bicycle riding

Behind Eq. (1) is the idea that motor skill learning is concerned with the geometry of dynamics in $n$-space. That is, it involves a person or an animal becoming acquainted with the qualitative aspects of the time-dependent changes taking place in $n$ dimensions. Bernstein (1967, 1996) thought of the general problem of coordination as that of mastering the many DFs involved in a particular movement pattern – of reducing the number of independent variables to be controlled. In far-reaching respects, Eq. (1) and its conceptual backdrop address “Bernstein’s problem” (Turvey, 1977, 1990). Consider learning to ride a bicycle, specifically the basic challenge of keeping the frame of the bicycle and oneself upright. A bicycle has five main moving parts (handlebars, front wheel, crank–chain–rear wheel assembly and two pedals) resulting in 10 DFs (each part requires for its description one position and one velocity coordinate). The mathematician Ian Stewart (1989), p. 91, has remarked that, in order to ride a bicycle, “you must gain intuition about the motion of a point in 10-space.” The bicycle’s behavior can be represented by the time variations of a single (10 dimensional) point that defines all of the bicycle’s 10 variables simultaneously. At one level of description, the would-be-bicycle rider is confronted by all of the force and motion complexity of the bicycle and its parts (torques on handlebars, pumping motions of the pedals, and so on); at another level, she is confronted – perhaps – by one point. At this alternative level, the question becomes: Is there an equation of motion for this one point, an equation that expresses all possible motions of the point? If there is, then it would be the “intuition” that the would-be-bicycle rider must “gain.” The requisite intuition is both simplified and made more profound by (a) the possibility that the behavior of the 10 variables is fully accommodated by the behavior of one variable (or, at most, a few variables) composed in some way from the initial 10 but differing markedly from them in the relative slowness of its changes and (b) the likely presence of stabilities, that is, preferred locations and paths to which the behavior is drawn and to which it tends to return if momentarily displaced. Feature (a) expresses the idea of a collective variable or order parameter $\xi$ (Haken, 1977, 1983, 1996) that captures the relative variations of the bicycle’s 10 DFs and feature (b) expresses the geometry or essential qualitative aspects of the bicycle’s dynamics.

A motion equation such as Eq. (1) applied to bicycle riding is, therefore, a very special kind of motion equation. It expresses the time-evolution of a relation (rather than of a force or an energy kind). Its solutions (when the left-
hand side goes to zero) define the stable and unstable organizational or coordinate- nal states, that is, the attractors and repellors of the task of riding a bicycle. As Kelso (1994, 1995) has highlighted, when Eq. (1) is applied to a motor skill (such as bicycle riding), the collective variable $\xi$ is an information- al quantity specific to the coordination and Eq. (1) is the law $L$ that governs its time variation. The existence of dynamical laws of informational quantities, and their significance to the functioning of complex systems, has been noted by several authors (e.g., Davis, 1988; Kugler and Turvey, 1987, 1988; Rosen, 1985, 1986). The claim that a motion equation such as Eq. (1) is what is learned when one learns a skill, is a claim that motor learning is about informational quantities (e.g., relational collective variables) and the dynamical laws governing their behavior (see Kelso, 1995). An information or coordination law is usefully expressed as a vector field such as shown in Fig. 1. The vector based at each point in the state or phase–space depicted in Fig. 1 is a dynamical rule: The state must evolve with the speed and direction of the vector based there (Abraham and Shaw, 1987).

Representing a dynamical system in the manner of Fig. 1 is the legacy of Poincaré, 1892–1894. Confronted with any time-dependent physical system, but particularly one that is complex in either composition or behavior or both (e.g., Turvey, 1988; Yates, 1982), Poincaré’s strategy was to pursue a meaningful, geometric characterization of its evolving states. In response to situations that resisted understanding through Newton’s analytic methods, such as the three-body problem, Poincaré introduced qualitative methods. These methods, refined and extended by modern dynamicists, do not answer specific questions on the exact values of a motion equation’s solutions. What they do answer are questions about the long-term behavior of the solutions and for the study of complex systems that proves to be extremely useful and enlightening (Abraham and Shaw, 1992).

1.3. Productivity of coordination dynamics

The points in Fig. 1’s state space are virtual, that is, they are all of the states in which the system of bicycle and rider could find itself. Consequently, what the vector field of Fig. 1 makes clear is that a coordination law $L$ satisfies the fundamental requirement of generativity or productivity (see Fodor and Pylyshyn, 1988) – it is a prescription for how to behave in novel circumstances. If a coordination dynamic in the form of Eq. (1) characterizes what has been learned when one learns a coordination, then it means that one knows how to produce the coordination in (almost) all circumstances in
Fig. 1. (A) A dynamical system on a phase or state space consists of a vector assigned to each point with each point interpreted as a possible or virtual state of the system. The vectors are specifications of how the system should evolve - in direction and speed - from each virtual state. (B) The curved arrows represent the flow of the dynamic, that is, the simultaneous movement of all initial states along their trajectories. From any starting point or initial state, there is a uniquely determined curve generated by following the specifications at each point the curve passes. As can be seen, the curves are converging on a region of the phase or state space. This region is referred to as an attractor. [Fig. 1(B) is reprinted with permission from Amazeen (1996). Both are adapted from Abraham and Shaw (1987)].

which the coordination applies and not just those in which the coordination was originally acquired. Productivity refers to the fact that performance of a learned coordination outstrips the particulars of the experiences through which it was learned (see Weimar, 1973, 1977). The significance of accounting for productivity was well understood by Schmidt (1975) in his classical, pioneering work on schema theory (see also related concerns of Bartlett, 1932; Greene, 1975a, b, 1978; MacNeilage, 1970). That theory was dedicated to explaining how the experiences leading to skill acquisition could result in an ability to generate parameterizations of a skill (a "generalized motor program") suited to any circumstances into which the skill might happen to be
inserted. In short, schema theory was aimed at the apparent creativity of skilled behavior, that is, its generative, rule-like quality. This quality is intrinsic to Eq. (1) as noted in Section 1.1.

1.4. Von Holst’s fish and the contrast between the dynamics of Newton and Poincaré

In order to bring the preceding notions into focus, we can turn to a fairly fundamental instance of coordination and take note of how it is characterized in the classical dynamics of Newton and the newer (nonlinear, qualitative) dynamics of Poincaré. The instance in question is the phase- and frequency-locking of the fins of Labrus and other similar fish investigated by von Holst (1939). Using the notions of maintenance tendency (the tendency for each individual fin oscillator to continue at its own preferred frequency) and magnet effect (the tendency of each fin oscillator to force its preferred frequency on the other), von Holst summarized the stable or absolute coordinations as those attainable when the maintenance tendencies were not too disparate and the magnet effect was strong. When the competition was equal to or greater than the cooperation (that is, the maintenance tendencies successfully countered the magnet effect), then the coordination was relative, with the rhythmic units moving at different tempos. Despite the absence of a fixed phase relation and a common frequency, fins in relative coordination nevertheless exhibited an attraction to some phase relations more than others. These attractive phase relations were those of absolute coordination. Thus, a change in the phase relation on each successive cycle, the so-called phase wandering, gave way to a hovering in the vicinity of the preferred pattern of absolute coordination for several cycles, which then gave way once again to phase wandering, and so on in a repeating pattern.

von Holst (1939) was able to address all of his major empirical findings on interfin coordination through a mechanical model of two oscillators coupled through a viscous medium. This model, which provides a Newtonian perspective on interfin coordination, is depicted in Fig. 2 (middle panel). The competition of maintenance tendencies is instantiated in the contrast between the uncoupled frequencies of the two oscillators. The magnet effect is instantiated in the effect of the oscillations of the large pendulum on the oscillations of the small rotating paddle, an effect that is scaled to the amplitude of the large pendulum’s motions. Transfers of momentum and energy define the nature of the coupling between the two oscillators. It is intuitively obvious that the system depicted in Fig. 2 (middle panel) is high-dimensional and that the
Newton’s Perspective (Ordinary Dynamics): \[ \begin{align*} \Phi_{\text{stable}} & \quad \Phi_{\text{unstable}} \\ \{ & \} = f (F_i, F_v, F_g, \ldots) \]

Poincaré’s Perspective (Coordination Dynamics): \[ \begin{align*} \Phi_{\text{stable}} & \quad \Phi_{\text{unstable}} \\ \{ & \} = f (\Phi; C) \]

Fig. 2. The middle panel is von Holst’s model of a mechanical coupling of two oscillators through a viscous medium. Adapted from von Holst (1937, 1939). See text for details.
relative phase of the two oscillators is the consequence of a very complicated interaction among inertial, gravitational, frictional, and elastic forces. The upper panel of Fig. 2 summarizes the shifting stable phase relations that would be observed as the amplitude of the pendulum oscillator, and the difference between the uncoupled frequencies of the two oscillators, change. The stationary relative phase – the stable difference between the phase angles of the two coupled oscillators – expresses the particular spatial and temporal aspects of the two oscillations at which a balance is achieved among the aforementioned forces.

The amplitude of the oscillator designated \( W_p \) determines the coupling manipulation – the larger the amplitude the stronger the influence of \( W_p \) on the rotations of the paddle O. The preferred frequencies in isolation of the two oscillators are designated by \( \omega_1 \) and \( \omega_2 \). Changing either the size or the shape of \( C \) brings about changes in \( \omega_2 \). The upper panel of Fig. 2 shows the changes in equilibria of the coupled mechanical oscillators as measured by the stationary values of the difference between their phase angles, \( \theta_1 - \theta_2 \). The changes depend multiplicatively on competition (the difference between the maintenance tendencies, \( \omega_1 - \omega_2 \)) and the strength of the magnet effect or cooperation. The lower panel shows how this complex mechanical situation can be addressed – through an equation based on the mechanical forces \( F \) (inertial \( (i) \), viscous \( (v) \), gravitational \( (g) \), and so on) or through an equation in the relational or collective variable of relative phase, such as Eq. (I), with \( C \) referring in the lower panel to the control parameters (and not to the spindle in von Holst’s diagram, middle panel).

Confronted with the dynamics of a system such as that of Fig. 2 (middle panel), Poincaré might have asked whether a simplification was possible: What does the system’s behavior look like geometrically? Or, relatedly, what does the system’s behavior look like in the long run, after the transients have died out? In a phrase, the system’s long-term behavior selects a simpler set of motions from among those that it is fully able to express. In modern terminology these simpler motions are related to attractors – portions of the \( n \)-dimensional space to which nearby points converge. What seems attractive in the behavior of the system depicted in Fig. 2 (middle panel)? A reasonable answer, already suggested, is “certain values of the cycle-average relative phase,” that is, the average difference during each cycle between the phase angles of the two oscillators. Consequently, in the perspective on dynamics that originated with Poincaré, the dynamics of the system depicted in Fig. 2 (middle panel) is captured most insightfully by (a) a kind of chart identifying which values of relative phase are attractive and which are repulsive for a giv-
en parameterization of the system and (b) a means of transforming the chart so as to identify the layout of attractive and repulsive values of relative phase as the parameters change. The chart is a vector field such as Fig. 1, drawn within a state space and expressing all the qualitatively different trajectories of the system at a given value of the control parameter $c$. Together, the state space containing the chart, and the transformations of the chart, comprise a state-control space (Jackson, 1989). An example of a state-control space is displayed in Fig. 3. The example is of a Hopf bifurcation – a transition from a point attractor in the phase plane to an orbital, limit cycle attractor in the phase plane as the control parameter changes (in short, the creation of a pe-

![Diagram](image)

Fig. 3. A phase-control or state-control space. As the control parameter increases from phase plane i to phase plane vi, the attractor bifurcates from a point to a closed orbit or limit cycle (A Hopf bifurcation).
rionic motion from a steady state). Armed with this chart and its transformations (that is, with the appropriate state-control space), our focus is deflected from the analytic and numerical details of the forces and motions of the system shown in Fig. 2 and drawn toward the global qualitative characteristics of \( \phi \)'s behavior. The obvious question is whether, so armed and so drawn, we can express and exceed the ability of von Holst's mechanical system to capture the interfin coordinations of \textit{Labrus}. Applying the Poincaréan strategy we have arrived at a surprising question: Are the interesting phenomena exhibited by the system depicted in Fig. 2 (middle panel) to be found in an equation of the time-evolution ("motion") of a single macroscopic relation an equation such as Eq. (1)?

1.5. The "unreasonable effectiveness" of Poincaré's strategy

Haken et al. (1985) applied the Poincaréan strategy to the coordination patterns that intrigued von Holst. That is, rather than dealing with the time-evolution of forces and physical configurations characterizing the system shown in Fig. 2, they focused simply on the time-evolution of what appears to be the key, single relation – the phase of one oscillator relative to the phase of the other. More particularly, they focused on what this relation does in the long run, its asymptotic behavior: For two limbs or two fins sharing a common frequency, relative phase gravitates to either 0 or \( \pi \), that is, to in-phase or anti-phase. This attraction of relative phase to these values is the major geometric feature of the dynamics of two interacting limbs or fins. They are points in the chart to which nearby points converge. Another important geometric fact emerges when the interaction is made to go faster and faster (Kelso, 1984). The attractive point at \( \phi = \pi \) becomes a repulsive point, one from which nearby points diverge, and only the attractive point at \( \phi = 0 \) remains. Movement speed or rate transforms the chart.

The motion equation of Haken et al., and the steps in its development based on the preceding geometric features, are widely known, as are the equation's extensions (Kelso et al., 1990; Schöner et al., 1986). It will, nonetheless, prove helpful to review them here. They make apparent Poincaré's legacy and its "unreasonable effectiveness." The latter phrase, used in relation to mathematics and its ability to capture physical laws of nature's regularities, is that of Wigner (1970). The application of the phrase to Poincaré's legacy – his strategy of emphasizing qualitative rather than quantitative questions about dynamics – is in respect to its ability to both capture and predict the regularities of biological coordination. Despite the large number of (neu-
ral, muscular) subsystems involved in interlimb rhythmic coordination, and the rich complexity and variety of their individual functions, Poincaré’s strategy systematically reveals both the commonplace and unintuited (but correct) features of their collective behavior.

Returning to Eq. (1), let the order parameter $\xi$ be identified with relative phase symbolized as $\phi$. An important feature of Eq. (1) is that it can be put into the form

$$\dot{\phi} = -\frac{dV}{d\phi},$$

(2)

where $V = V(\phi, \omega)$ is a potential function (Gilmore, 1981; Haken, 1977; Jackson, 1989). A function such as $V$ is termed “potential” because $V$ is always decreasing along the solution curves of the differential equation $d\phi/dt = f(\phi)$ (Hale and Kocak, 1991; Strogatz, 1994). Accordingly, a system governed by a potential tends to damp asymptotically to a time-independent state, that is, to a point attractor. Eq. (1), therefore, is the motion equation of a kind of system referred to generically as a gradient system (Abraham and Shaw, 1992; Gilmore, 1981; Thom, 1969). Given the point attractors of the interlimb dynamics, Haken et al. postulated a potential function $V$ with the following properties: (i) it was periodic, that is, $V(\phi + 2\pi) = V(\phi)$, (ii) it was symmetrical, meaning that $V(\phi) = V(-\phi)$, and (iii) it could be developed as the Fourier series in, minimally, the first three $(n = 0, 1, 2)$ cosine terms (that is, even terms) as required by the coexistence of point attractors at $\phi = 0$ and $\phi = \pi$ and the inequality of their attractiveness. Hence,

$$V = -a \cos(\phi) - b \cos(2\phi).$$

(3)

The minus signs permit the portrayal of the potential $V$ as a landscape with the attractors at the valley bottoms (the minima) for positive $a$ and $b$. On this landscape, the behavior of two interacting limbs, or fins, or mechanical oscillators (Fig. 2) is given by Eq. (2). Representing the coupled system by $\phi$, a simple image is of $\phi$ as the coordinate of a particle that moves on $V$ in an overdamped fashion – it travels towards the bottom of a potential well at a speed proportional to the well’s slope. An additional simple image is of $V$, the landscape, becoming less indented as the ratio of $b$ to $a$ decreases to the point where a valley at 0 remains but the valley at $\pi$ becomes a hillok. The control parameter $b/a$ determines the relative strengths of the two attractors and is related, inversely, to movement speed. The equation of the particle’s motion on $V$ is given by taking Eq. (3)’s derivative with respect to $\phi$ as suggested by (2):
\[ \dot{\phi} = -a \sin(\phi) - 2b \sin(2\phi). \]  

(4)

The state space of this system is as simple as it gets, the real line. The solutions of Eq. (4) involving different initial conditions (starting values of \( \phi \)), and fixed \( a \) and \( b \), generate a vector field on the real line – its so-called phase portrait. Surprisingly, perhaps, but this is our chart for the 1:1 frequency locking between limbs, fins, and the mechanical oscillators of Fig. 2.

The chart can be transformed by tuning \( \Omega/s \), corresponding, as noted, to changes in movement rate. It can be transformed in an additional way. If the two limbs, fins, or mechanical oscillators behave differently in isolation (for example, left to itself, one oscillator might tend to cycle faster than the other), then when they are brought together at a single common frequency, neither perfect inphase nor perfect antiphase can occur (Turvey et al., 1986). That is, the points of convergence in the chart are shifted, by this asymmetry, away from 0 and away from \( \pi \) (the attractors for symmetrical components) according to the magnitude of the asymmetry. For the most general case, two segments locked together at the same frequency will not be dynamically identical and cannot, therefore, contribute to the collective dynamics in exactly the same way. In the mechanical system depicted in Fig. 2, the physical asymmetry between the swinging bath and the rotating paddle is obviously of relevance to the relative phase at which they settle, as shown in the insert. Returning to the potential landscape of Eq. (3), its symmetry of \( V(\phi) = V(-\phi) \) about \( \phi = 0 \) can easily be broken by adding a term. This symmetry breaking term displaces the point attractors from 0 and \( \pi \), for fixed \( a \) and \( b \), in a direction that depends on its sign. Some relatively simple considerations about coupling two oscillators equate this symmetry breaking term, in the form appropriate to the motion Eq. (4), with the difference, \( \Delta \omega = (\omega_2 - \omega_1) \), between the two uncoupled frequencies (e.g., Kelso et al., 1990).

We can now show fully the chart (the vector field on the line) and its transformations (changes in the control parameter, changes in the symmetry breaking term). Within this state-control space, shown in Fig. 4, the succession of phase portraits is shaped by the changing fixed points or equilibria \( \phi^* \) of the coordination. For any given values of \( \Omega/s \) and \( \Delta \omega \), the \( \phi^* \) are determined by noting the value of \( \phi \) at which \( \dot{\phi} = 0 \) (alternatively, returning to Eq. (3), it is a matter of taking note of the point in the landscape at which the particle \( \phi \) comes to rest). Stable equilibria (attractors) and unstable equilibria (repellors) are determined by taking the derivative of \( \dot{\phi} \) with respect to
Fig. 4. Evolution of subcritical pitchfork (a1–a5) and saddle node or tangent (b1–b5) bifurcations with decreasing $bla$. The subcritical pitchfork bifurcation changes the attractor at $\pi$ to a repeller, leaving only the attractor at 0. The saddle node bifurcation is an initial coalescing of the unstable and stable states followed immediately by the annihilation of both leaving a saddle-node ghost that can continue to attract and repel. The dynamics are on the simplest phase or state space, that of the line. Panels a1 and b1 show the vector fields with inpointing arrows designating attractors and outpointing arrows designating repellors. Adapted from Mitra et al. (1997a).
ϕ evaluated at ϕ*. The resulting value, λ, is the characteristic value of the equilibrium or the Lyapunov exponent for the region near the equilibrium (e.g., Abraham and Shaw, 1992; Haken, 1983; Hilborn, 1994): λ < 0 for an attractor and λ > 0 for a repeller. The “attractiveness” of an equilibrium point is determined by the magnitude of λ < 0. The preceding technique involves the assumption that the system will not deviate substantially from linear behavior near the fixed points. This assumption was first applied by Poincaré in 1914 (Hayashi, 1964, p. 35).

The motion Eq. (4) with the added symmetry breaking term fulfills the deterministic part of Eq. (1), which for all intents and purposes may be all that is needed for a Poincaréan perspective on the mechanical oscillators of Fig. 2. With respect to the corresponding biological oscillators, however, the stochastic part, F(t), of Eq. (1) must be addressed. As a first approximation, this part can be interpreted as perturbations of ϕ welling up from a microscopic level (e.g., spontaneous firing of neurons) and equated with a Gaussian white noise process ζ, of strength Q > 0 (Schöner et al., 1986). Thus, we can write the complete rendition of Eq. (1) for the particular case of 1:1 frequency locking of biological components as

$$\dot{\phi} = \Delta \omega - a \sin(\phi) - 2b \sin(2\phi) + \sqrt{Q} \zeta.$$  \hspace{1cm} (5)

The added stochastic force influences ϕ as a function of λ. In the potential landscape, ϕ is displaced continuously from the bottom of a potential well by a random sequence of kicks resulting in a distribution of ϕ values concentrated at the value coinciding with the well’s minimum and shaped by the well’s concavity. The stationary probability distribution function P(ϕ; b/a) characterizing the momentary behavior of ϕ is derivable from the potential function (e.g., Gilmore, 1981)

$$P(\phi; b/a) = N e^{-(-a \cos \phi - b \cos 2\phi)/Q},$$  \hspace{1cm} (6)

where N is an appropriate normalization constant. The standard deviation of ϕ (SDϕ) around a point attractor, can be determined through Eq. (6) and λ of the point attractor (e.g., Gilmore, 1981; Schöner et al., 1986)

$$\text{SD} \phi = \sqrt{\frac{Q}{2|\lambda|}}.$$ \hspace{1cm} (7)

Given Eq. (7), a larger |λ| means a smaller variance in ϕ, and an interlimb system that is more likely to remain close to the fixed point attractor in the face of perturbations of strength Q.
In developing the above set of Eqs. (2)-(7), "an unreasonable" stance has been taken on the detailed microstructures of the central nervous system and the skeletomuscular system. Similarly, an unreasonable stance has been taken on the detailed inertial, elastic, viscous, and gravitational forces at play in the interacting mechanical oscillators of Fig. 2. The unreasonableleness lies in the degree of abstraction engendered by focusing on asymptotic, qualitative features of the macroscopic behavior of the respective biological and mechanical systems. An equation such as Eq. (5) is Poincaré's legacy. It puts into formal terms the observation made by von Holst that intersegmental coordination is a combination of competitive ($\Delta \omega$) and cooperative forces ($b/a$), an observation that von Holst was able to capture through the mechanical system of Fig. 2. The "unreasonable effectiveness" of Eq. (5) is evident in the following predictions regarding (a) the displacements of the steady-state coordinations $\phi^*$ from a required phase $\psi$ of either $0$ or $\pi$ and (b) fluctuations ($SD\phi \propto \lambda^{-1}$) around these steady states, for values of the movement rate $\omega_c$ ($\approx a/b$) at which the coupled limbs execute the coordination.

1. When $\Delta \omega = 0$, $\phi^* = \psi$.
2. When $\Delta \omega < 0$, $\phi^* - \psi < 0$; when $\Delta \omega > 0$, $\phi^* - \psi > 0$.
3. When $\Delta \omega \neq 0$, $|\phi^* - \psi|$ is greater for $\psi = \pi$ than for $\psi = 0$.
4. For a constant $\omega_c$, $|\phi^* - \psi|$ is larger for larger values of $|\Delta \omega|$.
5. For a given $\Delta \omega \neq 0$, $|\phi^* - \psi|$ is larger for larger values of $\omega_c$.
6. $SD\phi$ is larger for $\psi = \pi$ than for $\psi = 0$.
7. For a constant $\omega_c$, $SD\phi$ is larger for larger values of $|\Delta \omega|$.
8. $SD\phi$ is larger for larger values of $\omega_c$.

These eight predictions have been confirmed through experiments that manipulated $\psi$, $\Delta \omega$, and $\omega_c$ (see summaries in Amazeen et al., in press; Schmidt and Turvey, 1995; Turvey, 1994). The "unreasonable effectiveness" of Eq. (5) is expressed succinctly in Fig. 5, which shows the fit of actual coordination equilibria as functions of joint manipulations of symmetry breaking and movement rate to those expected from the coordination dynamic (Amazeen et al., 1996). The unreasonable effectiveness of Eq. (5) is further evident in the fact that these same predictions are made and confirmed for two limbs of two individuals that are coordinated by looking (Amazeen et al., 1995; Schmidt and Turvey, 1994). In the light of this experimental success, it needs to be underscored that Eq. (5) was essentially based on observations (Kelso, 1984) that merely suggested the relations covered by predictions 1 and 6.

Beyond the predictions Eq. (5) makes for stable coordination states, its "unreasonable effectiveness" carries over into coordination instabilities, spe-
specifically, to the transitions from patterns at or near $\phi = \pi$ to patterns at or near $\phi = 0$. When Eq. (5) is symmetrical ($\Delta \omega = 0$), a hard (discontinuous) phase transition from $\pi$ to $0$ is predicted as movement rate moves $b/a$ through its critical value (see Fig. 4, left panels). This has been confirmed for both within-person coordination and between-persons coordination (Schmidt et al., 1990; Scholz et al., 1986). In contrast, when Eq. (5) is asymmetrical ($\Delta \omega \neq 0$), a soft (continuous) phase transition from the vicinity of $\pi$ to the vicinity of $0$ is predicted (see Fig. 4, right panels) and has been confirmed for the within-person case (Mitra et al., 1997a).

2. Pattern dynamics and the active DF producing those dynamics

The coordination dynamics of Eq. (5) are at the level of observable, relational behavior (Kay, 1988). They are the dynamics of a particular organization in which two limbs or fins oscillate at a single, common frequency. What Eq. (5) says is that only one dynamical, effective, or active DF is needed to characterize the dynamics of the interlimb pattern. Generally speaking, the number of active DFs is the minimal number of first order, autonomous, dif-
ferential equations required to capture fully a system's time-evolving behavior (Abarbanel, 1996).  

Active DFs are not to be equated with biomechanical DFs. In the various instances of interlimb rhythmic coordination to which Eq. (5) applies, the number of biomechanical DFs is obviously not equal to one. A typical experimental instance of this coordination (e.g., Sternad et al., 1996) entails a person oscillating two pendulums, one in each hand, by motions about the wrists in planes parallel to the body's sagittal plane, that is, a situation of (minimally) two biomechanical DFs. The biomechanical DFs in the movement of a body segment are determined by the number of perpendicular planes of motion it can take through an articulated surface. The shoulder, for example, has three perpendicular planes of motion through the glenohumeral articulation, and with respect to the linked segments of the arm participating in throwing, 17 biomechanical DFs are involved from the shoulder to the finger tips (Higgins, 1977). The active DFs at the level of the throwing pattern, however, will not be 17. The expectation, following from the considerations leading to Eq. (5), is that their number will be considerably smaller and may well be equal to one.  

---

1 Consider the familiar damped, sinusoidally driven pendulum (with low-amplitude natural angular frequency of unity). Regardless of the complexity of the pendulum's physical construction (see, for example, Baken and Gollub, 1990, p. 137), its essential space-time behavior can be described by the dimensionless second-order equation

\[ \ddot{\theta} + \left( \frac{1}{q} \right) \dot{\theta} + g \cos(\omega_D t) = 0, \]

where \( \theta \) is angle (with the double and single overdots signifying second and first time-derivatives, respectively), \( q \) is the damping parameter, \( g \) the forcing amplitude (not gravitational acceleration), and \( \omega_D \) the frequency of the driver. The number of dynamical DFs of this system is given by the smallest number of first-order, autonomous equations required to describe the system's behavior. The above well-known, second-order equation can be re-written as a system of first-order equations as follows:

\[ \dot{\omega} = -\left( \frac{1}{q} \right) \omega - g \sin \theta + g \cos \phi, \quad \dot{\theta} = \omega, \quad \dot{\phi} = \omega_D. \]

The phase of the driver is introduced as a new variable \( \phi \), giving a total of three variables (\( \omega, \theta, \phi \)). These three variables are then the dynamical DFs of the system, while (\( g, \omega_D, \phi \)) are the parameters governing its run-time behavior.

2 Faced with the structural complexity of the arm, the arguments from Section 1 suggest that a would-be-skilled thrower discovers a collective variable or order parameter \( \xi \) and one or more control parameters. The restriction on the latter is that any one of them suffices to move \( \xi \) through all of its qualitative variations. It might be conjectured (e.g., Kelso, 1994) that what the learner does at the level of the throwing movement itself is discover a dynamical system of co-dimension 1 (e.g., Guckenheimer and Holmes, 1983; Strogatz, 1994). Eq. (5) is such a system. It is a first order, autonomous differential equation in which either tuning \( b \) or \( \Delta \omega \) brings about bifurcations, that is, changes in the number and/or type of equilibria.
A learner’s discovery of a collective variable at the relational level dramatically reduces the active DFs of the pattern that constitutes a skilled action but it does not necessarily reduce the number of biomechanical DFs. To the contrary, skilled behavior is often characterized by the harmonious, simultaneous activity of many biomechanical DFs, far more than were involved in the unskilled productions of the behavior. When Bernstein (1967, p. 127) describes the general problem of coordination as the mastery of “redundant DFs within a kinematic chain of movement,” the mastery lies not in prohibiting biomechanical DFs but in reducing the number of variables arising in these linked biomechanical DFs that must be controlled independently. In the present terminology, the mastery lies in reducing the number of active DFs.

2.1. Coordinative structures and active DFs

With respect to Eq. (5), the pattern $\phi$ is a collective variable that informs the underlying subsystems most directly responsible for the pattern $\phi$ and its behavior. This is the peculiar circularity so often noted in discussions of complex systems such as biological movement systems (Haken, 1983, 1996; Kugler and Turvey, 1987, 1988). It gives rise to the question: How many independent subsystems are ordered directly by the collective variable $\phi$ and its dynamics as given by Eq. (5) or, synonymously, what is the minimal number of active DFs needed to produce $\phi$ and its dynamics?

The nature of the preceding question is conveyed through Fig. 6 which expresses a fundamental understanding of synergetics with regard to pattern formation in complex systems (Haken, 1983, 1996). What Fig. 6 makes clear is that $\xi$ is formed from the cooperative activity of a number of subsystems, which yield a set $q$ of collective variables $q_i$ ($i = 1, \ldots, n$), with the dynamics of each collective variable described in the form of a first-order, autonomous, ordinary differential equation – an active DF. Each $q_i$ is collective only with respect to the mechanical, muscular, and nervous components of the one or few biomechanical DFs whose dynamics it subsumes, but not with respect to the coordination pattern of concern (Schöner, 1994). The coordination pattern, that is, the dynamics of $\xi$, is manifest as a result of coupling the dynamics of these collective variables $q_i$. We will call the set of subsystems that generate the variables in $q$, together with the dynamics of $\xi$ that enslaves the behavior of the subsystems, a coordinative structure. The use of this term here extends and refines an earlier usage that referred to a group of muscles often spanning several joints that is constrained to act as a single functional
unit (e.g., Turvey et al., 1978). The unitary aspect is \( \xi \), the group aspect is \( q \). As defined here, the concept of coordinative structure is continuous with a somewhat more recent, three-tiered interpretation advanced by Kugler and Turvey (1987, Ch. 9) and Turvey et al. (1986). The bottom tier consists of the subsystems (atomisms), the top tier consists of the boundary conditions (intentions, instructions), and the middle tier consists of the coordination pattern (the cooperativity). The middle tier is formed from and, at the same time, is the source of order for the bottom tier.

Given the present definition of a coordinative structure, we can rephrase the question above in more general form as follows: (a) How many \( q_i \) compose a coordinative structure, that is, produce the dynamics of its relational variable \( \xi \)? and (b) How does this number change in the course of learning?

2.2. Acquiring a novel skill: From high-dimensional to low-dimensional control

In learning a complicated motor skill, the individual, either strictly on the basis of his or her own efforts or with the help of an instructor, gets acquainted with the essential nature of the coordination pattern. This is Fitts’ (Fitts, 1964) and Bernstein’s (Bernstein, 1996) early phase of motor skill learning, involving a great deal of explicit analyses, consciously introduced changes
in details, and dedicated attention. Accuracy and adequacy in establishing the skill's form, rather than speed and force, are this phase's ultimate criteria (Bernstein, 1996). During this early phase, the set $q$ is likely to be large and variable in contained elements. No matter what the number of biomechanical DFs the person feels comfortable using, there will be an uncertain number of subsystems for their control as the person struggles with establishing the lower level dynamical variables $q_i$ that produce the required coordination dynamics. Exacerbating the determination of $q$ is the obvious difficulty of resolving $\xi$. In the early phase of learning a fairly complicated skill, there will be a number of competing collective variables $\xi$ at the pattern dynamics level (see, for example, Haken, 1996, Ch. 12). As learning progresses, a few become ascendent and eventually, perhaps, just one suffices (Haken, 1996).

For the neophyte, therefore, control of the pattern is likely to be high dimensional, meaning that it is achieved through many subsystems yielding dynamical variables that are not yet collected or unitized, giving the impression (to both participant and spectator alike) that the behavior is "clumsy," "inaccurate," "inelegant," and so on. In slightly more technical terms the behavior is noisy, especially over successive executions where aspects of the pattern exhibited in one attempt are either not present or noticeably altered in the following attempt. Relatedly, we might say that the neophyte's behavior exhibits a degree of randomness in the sense that some of perhaps many candidate active DFs seem to be brought into play as chance events from one attempt to the next. In light of this, motor learning must then proceed through reducing, standardizing and stabilizing the dynamical variables yielded by the subsystems that generate the coordination pattern and then come to be enslaved (or ordered) by it. Once $\xi$ has been discovered, aspects of randomness in the pattern due to the large size of, and sheer variation in, $q$ must be diminished. This process is the intermediate phase of Fitts (1964) and second phase (standardization and stabilization) of Bernstein (1996, p. 235).

In contemporary applications of the Poincaréan strategy to dynamical systems, the notions of noise or randomness are treated circumspectly. The method of dynamical modeling consists of identifying the key variables at the heart of a behavior, and then rendering the critical aspects of its time-unfolding as the dynamics of the identified variables. More often than not, even the best such models of natural phenomena fail to elicit all the richness of the original process. This can be due to two reasons. First, the dynamics ascribed to one or more of the identified variables is either not correct or just not powerful enough. Second, some variables involved in the actual behavior are either missing from the model or have been misidentified. It is only in recent
years that the full power of nonlinear dynamical systems in modeling extremely complex behaviors with relatively few variables has been recognized. The classical strategy taken in many cases of complex or seemingly irregular behavior has been to assume that very many unidentified variables, often with fast time-scale dynamics, are participating in the behavior, making it "noisy" with respect to the gross pattern under observation. Specific models of these noise processes have been developed (e.g., Gaussian noise), and applied successfully to dynamical modeling in the form of stochastic differential equations. As Farmer and Sidorowich (1989) have cautioned, it is important to recognize that thinking of noise processes as being the cause of random-looking aspects of any natural behavior can be misleading. Stochastic methods comprise a mathematical technique for coping with inadequate information about relevant variables or lack of richness in the modeled dynamics; noise processes are models themselves, not physical reality. In a contemporary view, when noise processes are applied to the time variations of a dynamical variable (i.e., a time series), they are understood to stand in as models for the statistical behavior of very many, or for all practical purposes, indefinitely many, variables (dimensions, coordinates, or dynamical DF) that would be needed to describe the phenomenon under study in a deterministic fashion.

With respect to intermediate motor learning, exploring standardization and stabilization in dynamical terms, requires coming to terms with some subtleties in the differences between deterministic and statistical approaches to complex behavior. In the analyses that are developed and applied in the next section, intermediate learning emerges as a process of building low-dimensional determinism into the coordinative structure, or, of reducing \( q \) to a set of a minimal number of variables that completely capture the dynamics under study. From the practical, or experimental, point of view, the preceding means that the decline in variability of a skilled performance (e.g., the standard deviation of \( \zeta \)) with practice goes hand-in-hand with a decline in the number of active DFs at the subsystems level. The experimental challenge taken up in the next section is that of tracking the size of the set \( q \). Pursuing the Poincaréan strategy, a geometric perspective is taken on the unfolding dynamics of a coordination. More particularly, from the geometry of a measured time series, information is extracted regarding the dynamical system at the source of the measurement. From this geometric viewpoint, it will be shown that intermediate learning emerges as a process of streamlining subsystems to yield as few variables as needed to produce the coordination pattern.
2.3. A simple motor skill: Interlimb rhythmic coordination at a relative phase of \(\pi/2\)

For simple skills, as Fitts (1964, p. 262) remarks, the earliest phase may be of very short duration, "covering only the time required to understand instructions, to complete a few preliminary trials, and to establish the proper cognitive set for the task." With simple skills, appropriate approximations to \(\xi\) and its dynamics (its motion equation) are arrived at quickly. In contrast, refining (standardizing and stabilizing) a simple skill can be a prolonged process.

A simple motor learning task that permits the examination of the above ideas is that of learning to produce 1:1 frequency locking of the two hands at a relative phase of 90° (Lee et al., 1995; Zanone and Kelso, 1992). Let the oscillations of the two hands be restricted to motions about the wrist joints in planes parallel to the body’s sagittal plane. The collective variable \(\xi = \phi = \pi/2\) of the pattern dynamics is unequivocal and readily identified by example or by instruction. The biomechanical DFs are essentially two in number – the planes of motion through the radiocarpal articulations of the two wrists (but with the possibility of some motion in planes parallel to the body’s frontal plane through the talofibular–tibular articulation). The collective variable of the required bimanual pattern dynamics, \(\phi = \pi/2\), defines how the two biomechanical DFs should relate. In order to achieve \(\phi = \pi/2\), the two biomechanical DFs, L (for the left) and R (for the right) must assume local, stable pattern dynamics specific to their respective roles. In this simple motor learning task, once an approximation to \(\phi = \pi/2\) has been established, the issue for the learner becomes refining the number and type of underlying dynamical variables, \(q_L\) and \(q_R\), that are required to move L and R on their respective trajectories (Fig. 6). This issue, for the student of motor learning, is a very technical issue of making very subtle processes quantifiable. Of interest are the geometric objects, or attractors, on which the trajectories generated by the L and R subsystems evolve, and the precise local properties of this evolution. The question the student of motor learning faces is twofold: How might the number of dimensions of these attractors be determined from observations of the learner’s behavior, and how might the changing number of active DFs produced by the learner to control his or her movement trajectories along these attractors, be quantified as learning proceeds? As Kugler and Turvey (1987), Ch. 7 underscore, for biological movement systems, there is a need to hold distinct the attractor associated with an action subsystem, and the particular means by which that attractor
is realized. The pattern dynamics of a coordination might exert substantial influence upon the basic geometry of the attractors of the subsystems, but achieving expert, economical motions on those attractors can be a separate matter, based on the degree of attunement to local, muscular–articular information specific to the attractor (Kugler and Turvey, 1987).

3. The global and local geometry of the intermediate phase of learning

The immediate task is to identify and apply methods for deriving from experimental observations of individual learning trials the global and local geometry of the intermediate phase of learning. Specifically it will be shown how an experimentalist might track $q_L$ and $q_R$ over the learning of a bimanual rhythmic coordination in which $\dot{\xi} = \phi = \pi/2$.

3.1. The experiment

The data to be reported are from a representative participant in Experiment 3 of Amazeen (1996). In this experiment, the representative participant along with the other participants learned to oscillate two pendulums, one in each hand, at a relative phase of $\pi/2$. The hand-held pendulums were aluminum rods, 0.50 m in length and 0.025 m in diameter, attached to a wooden handle 0.12 m long. A 50 g metal ring was attached to the bottom of each pendulum. Because the pendulums were identical, the symmetry breaking term, $\Delta \omega$ (see Section 1.5) was equal to 0 rad s$^{-1}$. The participant held the pendulums with palms centered on the wooden handles and with the wrists positioned at the end of the armrest of a specially designed chair (see figure of apparatus in Amazeen, 1996; for a close approximation, see Sternad et al., 1992, 1996). She was instructed to create as smooth and as continuous a trajectory as possible with the pendulum and to hold the pendulum firmly in the hand so as to guarantee rotation about the wrist rather than rotation about the finger joints. The participant was instructed to keep each pendular motion as parallel to the body’s sagittal plane as possible. Neither the hands nor the pendulums were viewed during the course of a trial. Movement trajectories of each pendulum were collected using a Sonic 3-Space Digitizer (SAC Corporation, Stratford, CT). The measures $\phi$ (averaged over the trial) and SD$\phi$ were calculated for each individual trial.

Prior to beginning the experiment, all participants were instructed to produce $\psi = -\pi/2$ in the following manner: "In this experiment you will be asked
to produce a relative phase of 90°, where your right pendulum will lead your left pendulum by one quarter of a cycle. Every time your right pendulum hits its forward-most peak, your left pendulum should be vertical and moving forward. Every time your right pendulum hits its backward-most peak, your left pendulum should be vertical and moving backward.” They were shown both a picture of the required phase relation and were given a demonstration by the experimenter. The representative participant, like the other participants, practiced for approximately 5 min until she felt that she could control the pendulums. She was permitted to elect a comfortable frequency and to indicate when she was ready for data collection in each 30 s trial. Three experimental sessions (30 trials each; approximately 45 min each day) were performed on three consecutive days, for a total of 90 trials. None of the trials was repeated. The participant was not given any feedback regarding her performance during any of the trials and, as was the case for all participants, she was discouraged from practicing between experimental sessions.

In Sections 3.2–3.5 the details of the techniques for determining the global and local geometry of the π/2 coordinative structure (for simplicity, the negative sign will be omitted from hereon; see Fig. 9) and their evolution during learning are presented together with the underlying theory. In Sections 3.6 and 3.7, the techniques and theory are applied to the experimental data.

3.2. Intersection of trajectories and projections of phase-space

Consider the ideal undamped (friction-free), unforced oscillator. Its behavior is expressed in a coordinate system of position and velocity. The behavior is deterministic in this coordinate system because the dynamics written over the two identified variables prescribe completely the possible motion. The geometric consequence of this determinism is that, over any given dynamical run (i.e., unfolding of motion under a specific set of initial conditions and parameter values), the trajectory traced out by the motion in the coordinate system does not intersect with itself. (If there were any intersections, that would mean the values of the variables at the point of intersection could change in more than one way. There are uniqueness theorems regarding solutions of autonomous equations proving that the orbits of such systems do not overlap with themselves in their true phase–spaces, see for example, Hilborn, 1994; Jackson, 1989).

Now, if a driver of some non-identical frequency, phase and energy was introduced, the oscillator’s behavior would lose its non-intersecting property in the original two-dimensional coordinate system. With the advent of the
driver, at least one new variable would have been introduced into the dynamics, necessitating three coordinates (see Footnote 1). If the unfolding trajectory of this now three-dimensional system were viewed in the original two-dimensional coordinate system, the picture obtained would be a projection of motion in three-space onto a two-dimensional surface. Motions along the dimension orthogonal to the two dimensions of the projection surface would not be resolved, and points on the trajectory that are far apart along the orthogonal dimension might lie close together on the projection plane. Thus, the view of the system on the projection plane would contain intersections, implying that the system is not deterministic. Also, this projected view of the dynamics of the actual system would contain false neighbors – points that lie close together on the projection plane that are far apart in the actual phase–space of the system. In fact, the intersections observed on the projection would, in this case, be exactly due to false neighborliness of points. The techniques of tracking active DFs draw heavily on the differences in the geometric properties of projection and false neighborliness between various candidate phase–spaces for a system under observation.

3.3. Reconstructing phase–space from experimental time series

In experimental settings, measurements are taken from a physical or biological system in the form of a series of values over time along a few dimensions of interest. Each measurement series of this kind is one-dimensional: it tracks the time variation of a single measured variable. Thus, in the case of the bimanual coordination at $\pi/2$, measurements of the position of a pendulum tip during the course of a trial would yield a one-dimensional time series $x(t)$ of displacements. From the geometric considerations motivated above, however, the space in which the dynamical structure of this moving body segment ought to be viewed is not that of these one-dimensional observations. The dynamics likely take place in a phase–space of higher dimension (i.e., the number of active DFs is greater than one), and the measurement provides only a projection of the full dynamics on the axis of the observed variable (Abarbanel, 1996, p. 4). A contemporary technique for recovering the dimensions lost in the measurement projection is known as phase–space reconstruction, where the measured time series is embedded in higher-dimensional spaces in an attempt to recover information about the original, unknown phase–space of the system.

The remarkable aspect of this method is that the embedding theorem (Casdagli et al., 1991; Eckmann and Ruelle, 1985; Takens, 1981) allows for a way
of embedding a measured scalar time series such as \( x(t) \) in a space of vectors \( y(t) \) whose coordinates are \([x(t), x(t + T\tau), x(t + 2T\tau) \ldots]\), where \( T \) is some integer multiple of the sampling period, and \( \tau \) is an appropriate time delay chosen by inspecting a non-linear correlation function called \textit{average mutual information} (Abarbanel, 1996; Abarbanel et al., 1993). The average mutual information (henceforth, AMI) function simply provides an estimate, over a set of measurements, of the amount of information (in bits) that can be learned about \( x(t + T\tau) \) from knowing the value of \( x(t + [T - 1]\tau) \). Since the coordinates of the reconstructed phase-space are composed of time lagged values of the measurement series \( x(t) \), the less the AMI at a particular time lag, the more effective (in eliciting new information about the original dynamics) is the addition of a measurement at that time lag as a coordinate of \( y(t) \). In practice, the value of time delay at which the first minimum of the AMI function occurs is chosen as \( \tau \) (see Fraser and Swinney, 1986). As it turns out, a reconstructed phase-space of vectors \( y(t) \) is related to the original phase-space by smooth, differentiable transformations, with the result that several fundamental invariants of the original dynamics are preserved in the reconstructed phase-space (see Abarbanel, 1996, p. 18). Thus, systematic study of the geometry of the dynamics in the reconstructed phase-space provides valuable information about the original, unknown dynamics of the system.

3.4. \textit{Unfolding the attractor by progressive elimination of false neighbors}

Once a set of scalar measurements \( x(t) \) has been obtained, the first task in phase-space reconstruction is to establish whether the dynamical system of which the measurements provide a projection is low dimensional. In the absence of direct information about the variables involved in the original dynamics, several possibilities need to be entertained. First, the system may be very high dimensional, such that no low-dimensional model of it will absorb any significant aspect of its behavior. Second, there may be a few variables that are responsible for most of the system's behavioral variations, with some additional, and relatively low-amplitude, high-dimensional aspects (e.g., measurement noise). Third, the system may be purely low dimensional, meaning that a set of first-order, autonomous, ordinary differential equations written over a few variables can capture its behavior, which, of course, may still look quite irregular and seemingly high dimensional.

Following the logic of Section 3.3, analysis proceeds by first determining a suitable time delay (using AMI) and then using it to embed the dynamics into
increasingly higher dimensional reconstructed phase-spaces. Consider, for example, the oscillation of one of the hands in the bimanual coordination task described above. Measurements $x(t)$ along the principal axis of motion, recorded from the tip of the pendulum, when viewed in one dimension (i.e., projected on a line), will cyclically traverse the region of the line that lies between the peak amplitude values. For a measurement series obtained over several cycles of the oscillation, a large number of points on the line will lie close to a given point not because they are dynamically related to it, but because of the severity of the projection. The process of unfolding the attractor, or the geometric entity in the appropriately dimensioned (and, in this case, reconstructed) phase-space on which the dynamics evolves, is, accordingly, that of adding time-delayed values of $x(t)$ as coordinates of $y(t)$ until false-neighborliness due to projection is completely eliminated. A glimpse of this process is shown in the four panels of Fig. 7. Fig. 7(a) shows a sample time series measured from the tip of the pendulum on the right hand of a subject performing the bimanual $\pi/2$ task. Taking these data as the scalar measurement series $x(t)$ and computing the AMI function yields a recommended time delay of 26 sampling intervals. (The sampling rate is 90 Hz.) Fig. 7(b) shows the series embedded in 2-space (the coordinates are $x(t)$ and $x(t + 26)$). Fig. 7(c) and (d) show embedding in 3-space (i.e., the third coordinate is $x(t + 52)$) from two different viewing angles. The viewing angle in Fig. 7(c) provides a projection of this 3-space that is similar to the projection obtained in the 2-space embedding shown in Fig. 7(b). The viewing angle in Fig. 7(d), however, provides a projection roughly orthogonal to the one in Fig. 7(c). The depth of the attractor along this added dimension revealed through 3-space embedding removes many false neighbors that remained in the 2-space embedding of Fig. 7(b).

The quantitative aspect of unfolding the attractor in the above manner involves the determination of global false nearest neighbors (henceforth, FNN) in any given embedding space. The essence of the procedure lies in taking sets of points nearest to every data point from a series $y(t)$ embedded in $d$-space, calculating the Euclidean distances in that space, and finding out whether these distances change substantially when the same calculations are made on $y(t)$ embedded in $d + 1$-space. If the changes in distance exceed an adopted threshold, then embedding in $d + 1$-space is taken to have removed false neighbors that remained in $d$-space embedding. For time series in which all regions of the attractor are well represented (which is the case for all data presented here), percent FNN calculations have been shown to be robust against changes in threshold settings, and the issue of optimum values for these set-
Fig. 7. A time series and its embeddings in two- and three-space. See text for details.
tings has itself received extensive study in the context of a variety of physical time series (Abarbanel, 1996).

When enough coordinates have been added to \( y(t) \), such that all FNN have been eliminated, adding further coordinates does not reveal any more about the dynamics of the system. The number of dimensions at which this occurs is called the embedding dimension \( d_E \). There are two points worth noting about \( d_E \). First, for a given dynamical system, the \( d_E \) that is calculated can change as a function of which projection of its full dynamics is captured by the measurement series. Thus, two different experimental variables taken from the same system may yield different values of \( d_E \). However, as discussed in Section 3.5, important invariants of the original system can be extracted by studying how the system evolves on the attractor as unfolded in \( d_E \)-space. Second, if the measurement series is a projection of a system that contains very high-dimensional dynamics (or is contaminated by measurement noise), unfolding the attractor in the manner above may never succeed in removing all FNNs. Noise is formally infinite dimensional. No matter how many dimensions are added to \( y(t) \), noisy \( x(t) \) will always wish to be unfolded in a larger dimension. For finite-precision numerical calculations on finite amounts of data, however, \( d_E \) analysis will fail to remove all FNNs only when the level of noise is significant as compared to the size of the attractor. Thus, in light of the discussion presented in Section 2.2, the results of \( d_E \) analysis have the advantage of indicating whether a coordination behavior assembled by a biological system contains low-dimensional, deterministic dynamics or high-dimensional dynamics. The latter might recommend stochastic methods for modelling, provided, of course, that the source of the noise is not of system-extrinsic, experimental origin. The hypothesis that has already been advanced is that, in the subsystems involved in a motor coordination task, the former is distilled from the latter during the process we call skill learning. Preliminary evidence to this effect is presented in Section 3.6.

3.5. Determining the number of active DFs

As just noted, the value of embedding dimension \( d_E \) that is obtained during phase-space reconstruction can change depending upon the choice of measurement variable. In general, if the dimension of the system's attractor in true phase-space is \( d_A \), then unfolding a measurement projection in an embedding space of \( d_E > 2d_A \) suffices to undo all intersections due to projection (Abarbanel, 1996, p. 19). (Note that, while \( d_A \) may be either a fractal or integer dimension, \( d_E \) is always an integer dimension.) However, since the dy-
namical system underlying the measurement remains the same no matter what one chooses to measure of its behavior, the number of active DFs (i.e., the number of first-order, autonomous, ordinary differential equations) of the system is invariant over changes in the measurement projection. The geometric interpretation of this invariant is the number of dimensions required to describe the local evolution of the orbit of the system around the attractor. The critical intuition is that this local dimension can be, and often is, less than the dimension of the space in which the attractor itself lives. Consider as an example an oscillator that has two irrationally related frequencies in its motion. The orbit of this system lies on a two-torus, a geometric object closely resembling a doughnut. While the number of dynamical DFs of the system is clearly two, the attractor itself would require $d_E = 3$ to unfold fully. The case of a hypothetical system that has an attractor shaped like a Möbius strip contains the same contrast — the local dimension of the system’s evolution is two, but the attractor lives in 3-space.

In our preferred method (see Abarbanel, 1996, for details) of determining the number of the active DFs, that is, the local dimension $d_L$, analysis begins in a working (reconstructed) space of dimension $d_W \geq d_E$ to ensure that the attractor is fully unfolded (i.e., all neighbors are true neighbors). Then, for any point in this space, the procedure tries to find a sub-space of dimension $d_L \leq d_E$ in which accurate local neighborhood-to-neighborhood maps of the data can be constructed. Neighborhoods of several sizes are specified by taking sets of $N$ neighbors of a given point $y(t)$, and then a local rule is abstracted for how these points evolve in one time-step into the same $N$ points in the neighborhood of $y(t + 1)$. The success rate of the rule is measured as percent bad predictions (henceforth, $\%bad$). The target of the analysis is to determine a value of $d_L$ at which $\%bad$ becomes independent of $d_L$ and of the number of neighbors $N$. This value of $d_L$ gives the number of active DFs of the system.

Besides indicating the number of variables that should be used in attempts to model the dynamics of a system, $d_L$ analysis also provides indications of local, low-amplitude (and/or fast time-scale) noise that may not be detectable in the attractor-level analysis of $d_E$. In Section 3.4 it was noted that $d_E$ analysis fails to remove all FNNs only when the level of noise at the scale of the attractor is high. In $d_L$ analysis, however, the value of $\%bad$ at which it becomes independent of $d_L$ and $N$ can serve as an indicator of noise at finer space–time scales. This feature of $d_L$ analysis comes into focus in the context of the data presented in Section 3.6.
3.6. Analysis of learning the π/2 coordination: An overview of the learning process

Turning now to the data, Fig. 8 shows samples of the data from the left and right hands of one representative participant. Panel A shows data from early in the second session, panel B from late in the second session, panel C from early in the third session, and panel D from late in the third session. These trials were chosen to mark their specific stage in the learning process because detailed analysis found them to be highly representative of their own neighborhood of trials. All phase–space reconstruction analyses presented below were applied to the same four trials for consistency and clarity of exposition.

Fig. 9 shows the subject’s learning profile with respect to pattern dynamics over the entire experiment. This figure plots average $\phi$ and its standard deviation $SD\phi$, each data point providing the average of the respective measure over the past five trials. On first inspection, the data on $\phi$ suggests roughly three stages of interest. In the first 30 trials or so, the subject’s phasing performance is poor, and appears not to make much progress toward the required value. A sharp improvement occurs during the next 15 or so trials (stage two), and then phasing performance does not change much over the rest of the trials (stage three). The $SD\phi$ profile mirrors the same three stages, but in a slightly different manner. During the first stage, $SD\phi$ decreases gradually even as $\phi$ performance does not improve much. During the second stage, it increases and then starts to decrease gradually as $\phi$ approaches the required value rapidly. During the third stage, as $\phi$ stabilizes, $SD\phi$ keeps decreasing initially, and then stabilizes toward the end of the experiment.

Close inspection of Fig. 8 reveals several interesting aspects of the process chronicled in Fig. 9. Panel A of Fig. 8 provides a window into the early part of the second stage just discussed. It shows a portion of the 34th trial of the experiment (the fourth of the second session). The irregularity of each pendulum’s velocity is obvious. On closer inspection, however, it is somewhat systematic and informative. In particular, each cycle of each pendulum contains regions of marked dips in velocity, and they do so with some synchrony. The task requires that the peak amplitude points of the right hand coincide with the zero amplitude points of the left hand. It is noteworthy that the dips in velocity of the right hand tend to occur slightly after the peak amplitude points, suggesting that the rotation of the right hand slows down just past its point of required synchronization with the left hand. The dips in velocity of the left hand, however, tend to occur around the zero amplitude locations,
Fig. 8. Samples of the data from the left and right hands of one representative participant in Experiment 3 of Amazeen (1996). Panel A shows data from early in the second session, Panel B from later in the second session, Panel C from early in the third session, and Panel D from late in the third session.
the left hand's point of synchronization with the right hand, and this tends to coincide approximately with the dip in the right hand's velocity. These dips in velocity around the synchronization points suggest an ongoing attempt on the part of the learner to develop attunement to the relevant relatio
tal quantity, \( \zeta = \phi = \pi/2 \), at the identified points of synchronization. It might be suggested, perhaps, that the learner slows the motions around synchronization points to develop a feel for where in their respective cycles the hands are (i.e., proprioception via the haptic perceptual system) as compared to where they ought to be (which is at this stage better recognized near synchronization points than anywhere else in the cycle).

Panel B of Fig. 8 shows a portion of the 54th trial of the experiment (24th of the second session). Examination of Fig. 9 places this trial at the end of stage two or the beginning of stage three, depending on whether one focuses on \( \phi \) or SD\( \phi \). With respect to \( \phi \), the third stage is well underway by the 54th trial in that no obvious improvements in phasing occur beyond this point of the experiment. With respect to SD\( \phi \), however, the 54th trial falls more toward the end of stage two with variability still gradually approaching its eventual asymptotic value. This last aspect of the trial is evident in the slight irregularity of the left pendulum's velocity, particularly in comparison to the
relatively smooth motion of the right hand. It may be worth noting that the point of synchronization discussed above occurs at maximum velocity for the left hand, while for the right hand it occurs at zero velocity. The slower rate of smoothing in the left hand’s motion might then be attributed to greater difficulty of proprioceptively registering its phasing status at the point of synchronization with the right hand.

Panel C of Fig. 8 shows a portion of the 65th trial of the experiment (the fifth of the third session) and panel D shows a portion of the 85th trial (the 25th of the third session). Inspection of Fig. 9 suggests that both these trials occur well-within stage three, when all discernible improvements in both \( \phi \) and \( \text{SD}\phi \) have ceased. Indeed, mirroring their near equivalence with respect to the operative relational quantity and its pattern of variability, the data in the two panels do not present any obvious differences to the eye. Following the discussion of the early stage of learning of Fitts (1964) and Bernstein (1996), one might note that the mechanisms of this stage are evidenced only in panels A and B of Fig. 8. The coordination dynamic, or the skill in question, has been acquired by the end of the second session, past which there is little evidence of improvement in performance at the level of the pattern being learned. This raises the question of what, if anything, of interest might be occurring in the period between panels C and D that is of interest to learning theorists, but not easily accessed through pattern-level, relational analysis. In particular, what might be said regarding Bernstein’s standardization and stabilization or Fitts’ intermediate stage of learning? These questions can now be addressed through the inspection of the dynamical behavior of the subsystems L and R during the course of learning. For this we turn to the results of the phase-space reconstruction analysis of the behavior of each hand.

3.7. Analysis of learning the \( \pi/2 \) coordination: Phase-space reconstruction

Figs. 10 and 11 present, respectively, the results of the \( d_E \) and \( d_L \) analyses on the same four trials shown in Fig. 8. Focusing on the \( d_E \) analysis first, we note that the rough division observed above between Panels A–B and Panels C–D of Fig. 8 are reflected on the same sets of panels in Fig. 10. Panels A and B of Fig. 10 show that, while the attractor on which the right pendulum travels unfolds completely in 4 dimensions, the attractor of the left pendulum fails to unfold completely in both cases. In Panel A, \( \%\text{FNN} \) for the L subsystem decreases rapidly with the addition of up to four dimensions during reconstruction, but then fails to fall to zero with further addition of dimensions. In panel B, this failure of \( \%\text{FNN} \) to reach zero is less dramatic,
Fig. 10. The results of the $d_0$ analysis conducted on the four trials shown in Fig. 8. See text for details.
Fig. 11. The results of the \( \delta \) analysis conducted on the four trials shown in Fig. 8. See text for details.
but present nonetheless. The failure of %FNN to reach zero indicates the presence of high-dimensional activity (noise) in the L subsystem’s dynamics at the scale of the attractor’s size. In panels C and D of Fig. 10, however, the attractors of both the L and R subsystems unfold completely in low-dimensional spaces, the final dimensions (as far as can be discerned) being 4 for R and 5 for L. Thus, the disappearance of high-dimensional activity (at the scale of attractor size) from the dynamics of the L subsystem parallels the onset of smooth velocities in the left pendulum’s motion as noted above (and shown in Fig. 8). This process of eliminating noise at the subsystem level also parallels the offset of the second and the onset of the third stage identified for pattern-level, relational dynamics (as evidenced in Fig. 9). Just as in the case of pattern-level analysis, however, $d_E$ analysis does not reveal any obvious changes in subsystem-level dynamics between Panels C and D. High-dimensional activity at the scale of attractor size appears to have been eliminated by the time pattern-level dynamics enters its stage three.

The subtler, but no less significant, process of standardization and stabilization becomes detectable only through inspection of each subsystem’s local evolution behavior as it moves on its respective attractor. For this, we turn to the results of $d_L$ analysis, shown in Fig. 11. Recall that the number of active (dynamical) DFs of a system, as revealed through $d_L$ analysis, is the value of the dimension at which %bad becomes independent of both the number of dimensions and neighborhood size. Recall also, that the value of %bad at which this occurs can serve as an indicator of fast time-scale and/or low-amplitude high-dimensional activity that may not be evidenced in analyses at the scale of attractor size. Focusing first on the R subsystem results in the four panels of Fig. 11, we note that %bad becomes independent of both neighborhood size and dimensions at about four dimensions. Note also that the value of %bad at which this occurs decreases steadily over the four panels, reaching a final value of zero (near-perfect prediction) in Panel D. The results of the L subsystem are similar, with the quality of predictions being uniformly worse than for the R subsystem. Prediction for the L subsystem is very poor in Panel A, and poor in panel B, with improvements in the last two panels ultimately taking it close to the final value for the R subsystem. The final reading for the number of active DFs for the L subsystem is five.

The difference between the number of active DFs obtained finally for the R subsystem (four) and L subsystem (five) should not be interpreted without further study. Close inspection of Figs. 10 and 11 can provide motivation for this reserve. Note, for instance, that $d_E$ for the R subsystem is 4 for panels A, B, and D, but 5 for panel C of Fig. 10; $d_L$ for the same subsystem, like-
wise, comes out to be 4 for panels A, B, and D, but 5 for panel C (Fig. 11). The precise determination of the reliability of such subtle differences requires statistical tests performed over results from multiple trials and subjects, and such studies are being reported elsewhere. Furthermore, it should be underscored that the L versus R asymmetry should not be hastily attributed to handedness either. The presented data are from a left-handed subject.

In any event, it suffices for the arguments being advanced in the present paper that Figs. 10 and 11 show the distillation during learning of low-dimensional, deterministic dynamics at the level of the subsystems that produce the coordination pattern. With respect to the low-dimensionality aspect of the claim, $d_E$ analysis revealed that, over learning, the attractor for each hand migrated from a space of high dimensions to one of at most five dimensions. Concomitantly, $d_L$ analysis suggested that, after acquisition of the skill, no more than five active DFs were required by each hand to produce the coordination dynamics of the bimanual pattern. With respect to the deterministic aspect of the claim, $d_E$ analysis revealed the early process of eliminating high-dimensional activity at the scale of attractor size, and $d_L$ analysis showed that this process continued locally on the attractor, at ever finer space-time scales, even after performance at the level of relational analysis (or, for that matter, dimensionality analysis at the scale of attractor size) had ceased to show discernible signs of continued learning.

4. The significance of the active (dynamical) DFs

A traditional view of motor control is that the nervous system's control variables are biomechanical variables – such as joint angle, movement amplitude, force or torque, stiffness, and so on. The nonlinear evolution rules governing the motions on the attractors of the left and right hands must be in respect to variables that are considerably more abstract than biomechanical variables and at some remove from these surface expressions of movement. Within neuroscience there is a growing sentiment for looking beyond conventional mechanics for the variables in which to describe and conceptualize the neural contribution to movement patterns (e.g., Feldman and Levin, 1995; Hasan, 1991; Latash, 1993; Loeb, 1987). What DFs do muscles have that could be specified reliably by the central nervous system? One major step toward an answer is the proposal that the sought-after neural control variables must satisfy a criterion of independence from external forces and biomechanical variables (Feldman, 1986; Feldman and Levin, 1995). The primary can-
didate in this regard is the variable known as $\lambda$, the threshold muscle length at which the autogenic recruitment of $\alpha$-motoneurons is initiated, selecting, thereby, a force-muscle length or torque-joint angle characteristic curve (e.g., Latash, 1993). A change in $\lambda$ results in parallel shifts in these curves along the muscle length or joint angle axis.

Feldman (1980) and Feldman and Latash (1982) have suggested that single joint control is through a pair of "commands," $r$ and $c$, that have direct relations to common physiological notions of reciprocal activation and coactivation (Granit, 1970), respectively. The $r$ command corresponds to unidirectional shifts in $\lambda_{\text{flexor}}$ and $\lambda_{\text{extensor}}$ resulting in a change of the joint angle without a change in joint compliance. The $c$ command corresponds to contradirectional shifts in $\lambda_{\text{flexor}}$ and $\lambda_{\text{extensor}}$ resulting in a change in joint compliance without a change in joint angle. The $r$ and $c$ commands are spatial. A third additional command $\mu$ has the dimension of time. It reflects the dynamical sensitivity of muscle spindle afferents. Given a $\lambda$ value for a given muscle, the threshold muscle length when the muscle undergoes stretching is $\lambda^*$ which can be linearly approximated by

$$\dot{\lambda}^* = \lambda - \mu \frac{dx}{dt} \tag{8}$$

(where $x$ is the current muscle length and velocity is positive for increases in muscle length.) The $\mu$ command specifies a kind of friction and, as such, can destabilize joint equilibria and foster orbital rather than point stability (Levin and Feldman, 1995).

Taken together, therefore, the $r$, $c$, and $\mu$ commands can lead to the rhythmic behavior at a wrist joint in the bimanual task analyzed in Section 3. For low frequency movements ($\leq 1$ Hz), it has been suggested that control is through discrete shifts in $\lambda_{\text{flexor}}$ and $\lambda_{\text{extensor}}$ leading to a smooth combination of discrete flexion and extension movements. At higher frequencies, joint compliance is fixed through a $c$ command at a level conditioned on the movement frequency with the $r$ command shifting back and forth within limits corresponding to amplitude (Feldman, 1980). One immediate question, therefore, is whether the active DFs isolated through the phase-space reconstruction procedures can be related to the control variables of $r$, $c$, and $\mu$.

Before questions of this sort can be addressed, a better understanding of the active DFs appears to be necessary. In particular, it should be noted that phase-space reconstruction analysis provides information about the number of DFs that are active in a system's time evolution, but does not provide any clue as to their physical (or physiological) identity. Models of the oscillatory
behavior studied in the hand-held pendulum task must be constructed from what is known (or appears reasonable) from the physics and physiology of the situation. Reconstruction analysis can aid that enterprise only by providing important constraints for such models. The first such constraint is, of course, the number of first-order, autonomous equations that is likely required to capture the evolution of the system on its attractor. From the analyses presented above (and in Amazeen, 1996), and from other studies (Mitra et al., 1997b) on subjects oscillating hand-held pendulums, it is becoming apparent that models of the system must involve between three and five active DFs, depending upon load-level (bimanual) coordination requirements, and other such factors. A sustained oscillatory behavior requires a minimum of two active DFs, position and velocity. Additional active DFs may emerge, however, if the influence of a driver is introduced, or if parameters such as friction and stiffness in the system's equation of motion come to have dynamics of their own. These are the most obvious possibilities for modeling suggested by the results of reconstruction analysis. A modeling study along these lines is currently being conducted.

The issue of how these active DFs may relate to the three control variables $r$, $c$, and $\mu$ will probably be better understood once efforts at modeling have made some progress. At this time, it is only possible to make reasonably informed conjectures. Of particular interest is the possible contribution of the $\mu$ command in the scheme of Feldman's suggestion (Feldman, 1980) that fast oscillatory behavior about a joint is accomplished through fixing joint compliance via a $c$ command, and then letting the $r$ command undergo periodic shifts. As already noted, a nonzero $\mu$ command introduces a friction term that plays a role in adjusting the centrally provided $\lambda$ as a function of muscle velocity. Even for the most controlled of experimental settings, actual muscle velocities depend not only on the values of central commands but also on execution-time perturbations of external and uncontrolled internal (peripheral) origin. In light of this, it may be useful to consider the $\mu$ command also as a way of subsuming uncontrolled (and possibly high-dimensional) execution-time exigencies into a low-dimensional dynamic at the expense of adding a single active DF in the form of friction dynamics. The process of eliminating high-dimensional activity from the orbits of the subsystems observed in the learning data presented in Section 3 may very well have its neurophysiological basis in this mechanism.

In studies recently conducted (Mitra et al., 1997b), but not reported here, subjects producing oscillations of a single hand-held pendulum (i.e., no bimanual coordination goal) of sufficient load yielded motions of only three
active DFs. Studies involving lighter pendulums, or bimanual coordination goals (as in this paper), yield subsystem motions of three to at most five active DFs. We conjecture that the three principal active DFs in single-joint movements are position, velocity, and a variable due to friction dynamics through the $\mu$ command. Additional active DFs may have their origin in systematically nonstationary values of the $c$ command (a possible stiffness dynamic), or other, as yet undetermined means (adopted, for example, in cases requiring bimanual coordination) of introducing the influence of another oscillator’s phasing, possibly in the form of a weak driver. As already remarked, it is difficult at this stage to evaluate these possibilities rigorously. Nonetheless, the close correspondence between the number of independent control variables postulated (for single joint movements) by Equilibrium Point theorists, and the number of active DFs yielded by single-joint movement subsystems uncovered through phase–space reconstruction, suggests that nontrivial connections between movement dynamics and movement control are likely to be found as research advances along these lines.

5. Concluding remarks and speculations

Motor learning can be considered as a pattern formation process within the roughly hewn frameworks advanced by Fitts (1964) and Bernstein (1996). That is, the formation occurs in phases – early, intermediate, and late – with a somewhat different emphasis in each phase. Using Fig. 6, it can be suggested that when learning is considered from the perspective of coordination dynamics, the early phase consists of discovering and establishing the relevant collective variable $\zeta$ that expresses the pattern, with the intermediate phase dedicated to refining the subsystems that both produce and are governed by $\zeta$. In both the early and intermediate phases, reducing active DFs is a key process. There may be a number of competing collective variables at the level of the coordination pattern (Haken, 1996) and there are certainly many candidate subsystems one level down functioning to produce the coordination pattern. Resolving one active DF or a few active DFs at the levels of pattern and subsystems (see Fig. 6) is, presumably, a major goal.

The consequence of attaining this goal is an action capability with the desired quality of a schema, namely, generativity. By the arguments advanced here and elsewhere (e.g., Amazeen, 1996; Kelso, 1995), learning endows the person or animal with an abstract dynamical capability of producing stable variants of the required coordination pattern that are suitably adapted to cir-
cumstances, both familiar and novel. This abstract nature of a learned coordination dynamic may bear on other classical issues in motor learning closely related to generativity such as motor equivalence. The difficulty here—in understanding how a pattern learned with one set of effectors can be generalized to others—is identifying the requisite type of abstraction. Clearly, the dynamics of the collective variable is appropriately abstract given that it is the dynamics of a spatial–temporal relation. Further suggestions of the requisite type of abstraction are provided by the above analyses of the global and local dynamics of the hands in bimanual rhythmic coordination revealing low-dimensional attractors and the active DFs by which the motions on these attractors are achieved. Considering the latter active DFs, they are not defined anatomically but functionally, in attractor-referential terms. Accordingly, acquiring a coordination dynamic means acquiring the entire structure depicted in Fig. 6, the abstract dynamics of both pattern and component subsystems, with a consequent capability to produce the coordination in relative independence from the effectors (see relevant discussions by Jordan, 1995; Keele et al., 1995).

Two further speculations may be warranted. One is with respect to the fact that a learned coordination exhibits flexibility and adaptability in addition to stability. The analysis of the attractors for the two hands reported in Section 3 (and Amazeen, 1996) revealed that they were not simply limit cycles plus high-dimensional noise as earlier research had suggested (Kay, 1988). To the contrary, they seemed to be strange attractors of definite dimensionality, opening up the possibility, therefore, of chaotic dynamics (see Mitra et al., 1997b). From the perspective of motor control, the possible significance of chaos is the facility of a synergy to explore wider regions of its phase–space and, thereby, take advantage of its large collection of potential states. Adaptation to perturbing influences is limited if a system is restricted to “an integrable subspace of the full system phase–space” (Abarbanel, 1996, p. vi). In short, if learning endows the relevant subsystems with deterministic randomness or persistent instability (that is, chaos), then it provides insurance that the acquired coordination can readily adjust, without loss of essential form, to the varied conditions in which the coordination is manifest (Amazeen, 1996).

The final speculation is with respect to what Fitts (1964) called the long term phase of motor learning. It has often been suggested that biological movement systems conform to a design in which the same coordination processes repeat from level to level. Although Sherrington's (Sherrington, 1906) principles of coordination—facilitation between agonistic actions, in-
hibition between antagonistic actions, and chaining – were originally formulated for a single level of function, contemporary observations might suggest that they apply, in some sense, to every level of functional organization, including the highest level at which motivations and intentions are formed (Gallistel, 1980). The maintenance tendencies and magnet effects that von Holst (1937, 1939) identified as the pattern forming processes at the level of appendages (e.g., the fins of Labrus) were recurrent, he felt, at the levels of the segments of the appendages (e.g., the finlets of a fin), the muscles governing the segments, and the cells governing the contractions of the muscles.

Invariance with respect to multiplicative changes of scale, or self-similarity, is essentially the notion expressed through homogeneous power functions (Schroeder, 1991). If learning a coordination entails the establishment of self-similar dynamics at each and every level and/or on each and every trial, then we may expect to find power functions expressing the learning process. It has long been known that there is a power law relation between the speed of performance on perceptual-motor skills and the number of practice trials (Fitts, 1964; Newell, 1991; Snoddy, 1926). Skills that are more aptly termed cognitive than motor have been shown similarly to satisfy a power law (e.g., Newell and Rosenbloom, 1981). A particularly significant expectation is that, given the possibility of indefinitely many scales at which the same coordination dynamics may be manifest, the effects of practicing a skill may never stop. The effects of practice can continue to reach down into the dynamics at even smaller length and time scales (Amazeen, 1996). This, it may be conjectured, was the case in respect to the classical power-law finding of Crossman (1959) which indicated that factory workers who had rolled 10 million cigars over seven years continued to improve, albeit at an ever decreasing rate.

The analysis presented in Section 3 and Amazeen (1996) gives some insight into the particulars of such a long-term process. As noted, whereas the $d_E$ analysis showed an early elimination of high-dimensional activity at the scale of attractor size, the $d_I$ analysis showed that this process continued locally on the attractor. The implication is that, once the relevant attractors for the pattern dynamics and subsystem dynamics have been established in the early and intermediate phases of learning, the high dimensional microfunctioning continues to be incorporated into the active DFs that determine the motion on the attractors and, thereby, the coordination. It may be conjectured that this latter process – of continually increasing the determinism of the acquired coordination – is Fitts’ (Fitts, 1964) late phase of learning.
Acknowledgements

This research was supported by National Science Foundation Grant SBR 94-22650 awarded to M.T. Turvey and a University of Connecticut Dissertation Fellowship awarded to P.G. Amazeen. The authors thank Henry Abarbanel, Elliot Saltzman, Richard Schmidt, Bruce Kay, Michael Riley, David Collins, Ramesh Balasubramaniam, and Claudia Carello.

References


