Working with Implicit Surfaces and Point Clouds

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Overview

- Motivation
- Geodesics and generalized geodesics
- Comparing point clouds
- Meshless geometric subdivision
- The future and concluding remarks

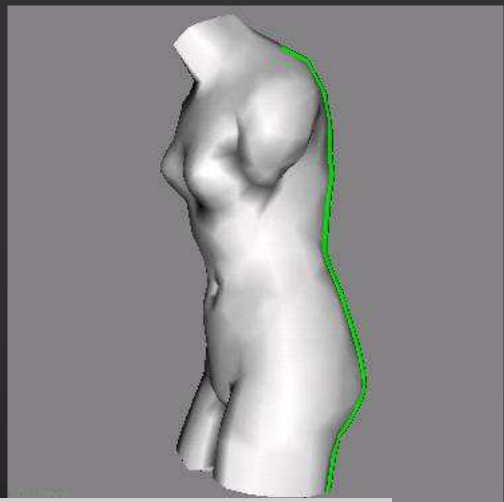
Motivation

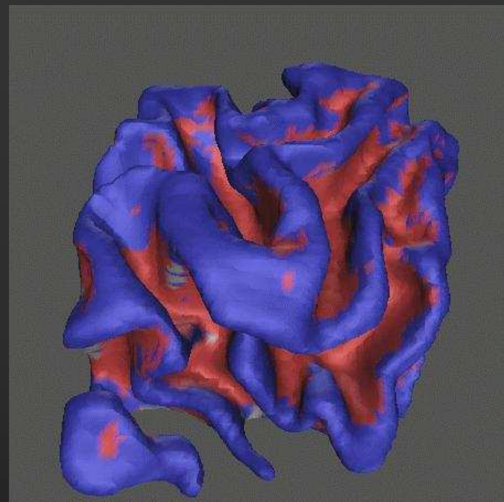
- **Implicit surfaces**
  - Facilitate fundamental computations
  - Natural representation for many algorithms (e.g., medical imaging)
  - Part of the computation very often (distance functions)
- **Point clouds**
  - Natural representation for 3D scanners
  - Natural representation for manifold learning
- **Dimensionality independent**
- **Pure geometry (no artificial meshes, etc)**

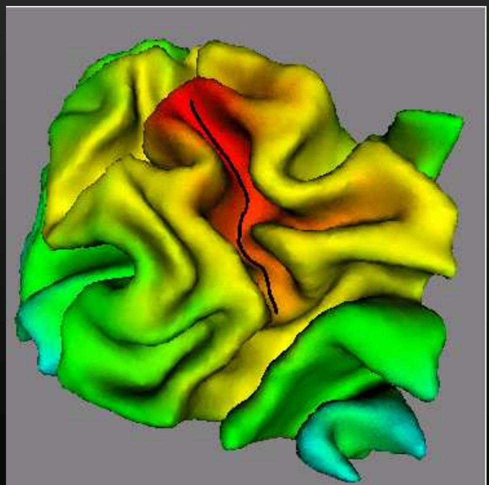
Geodesics and Generalized Geodesics

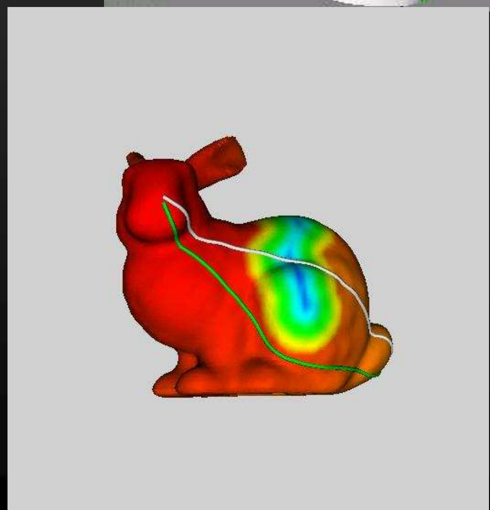
Joint with Facundo Memoli

Motivation: A Few Examples

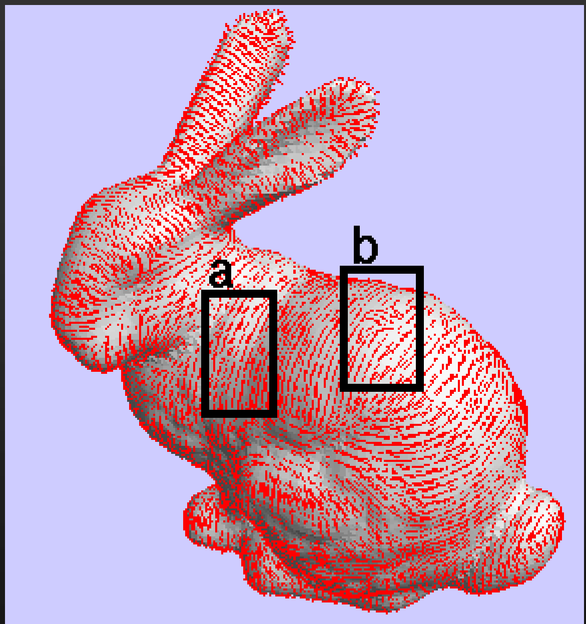




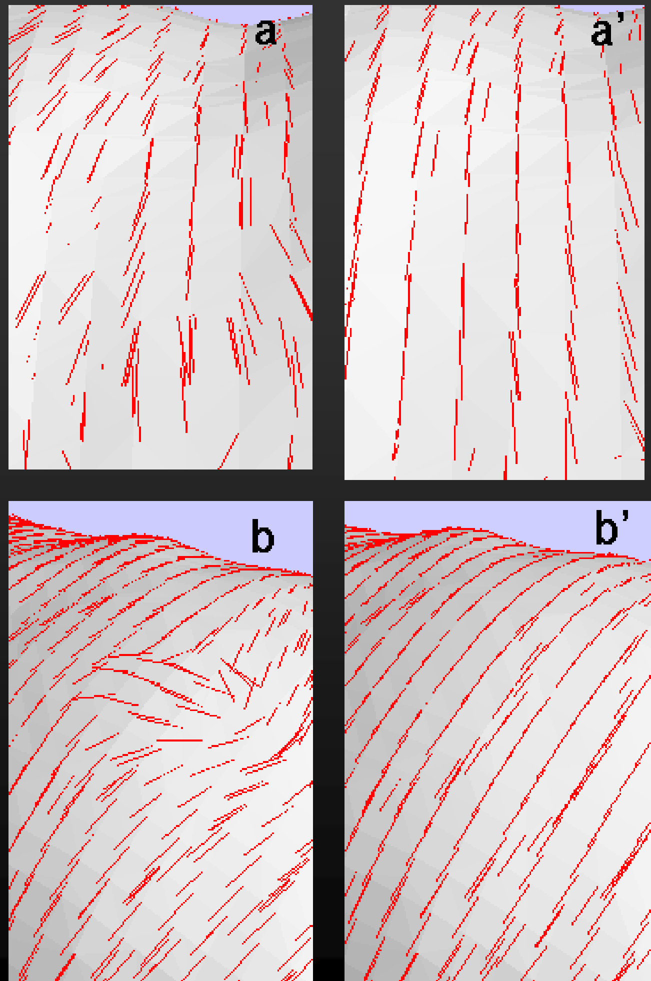




Motivation: A Few Examples (cont.)



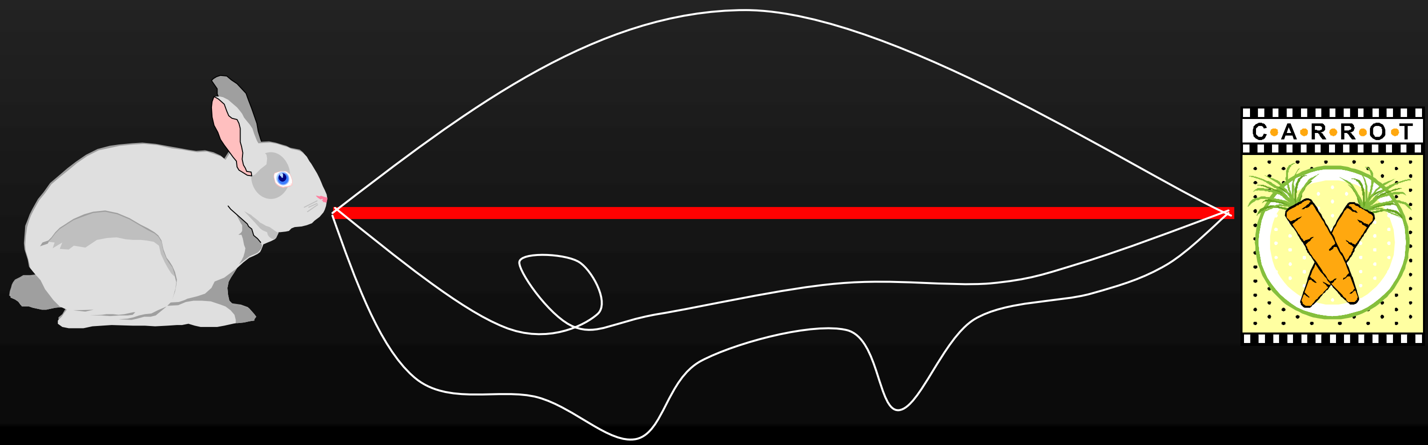
noisy



cleaned

Motivation: What is a Geodesic?

$$d_S^g(p,x) = \inf_C \int_p^x g(C)\,ds$$



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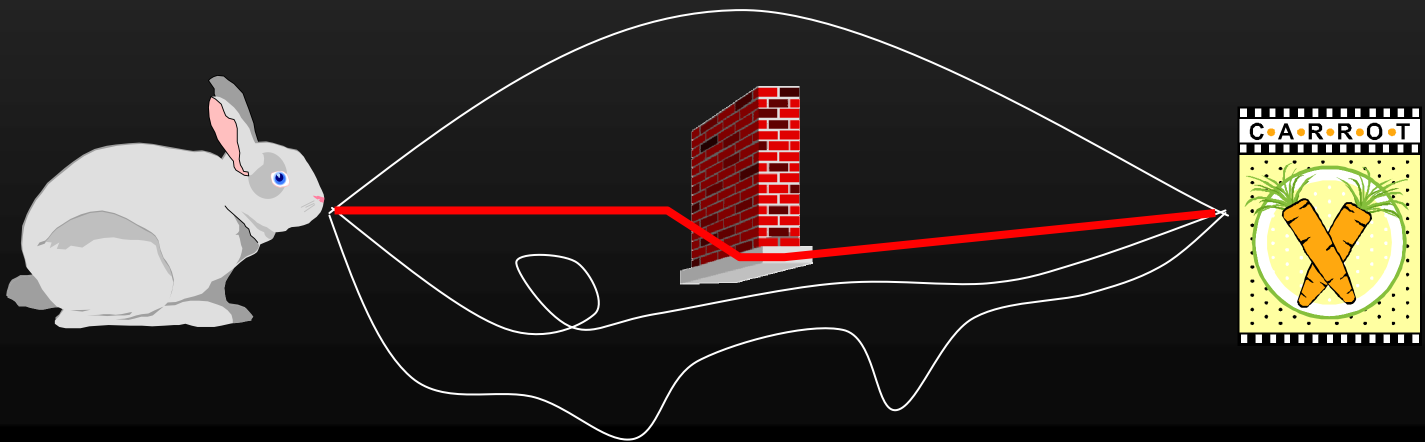
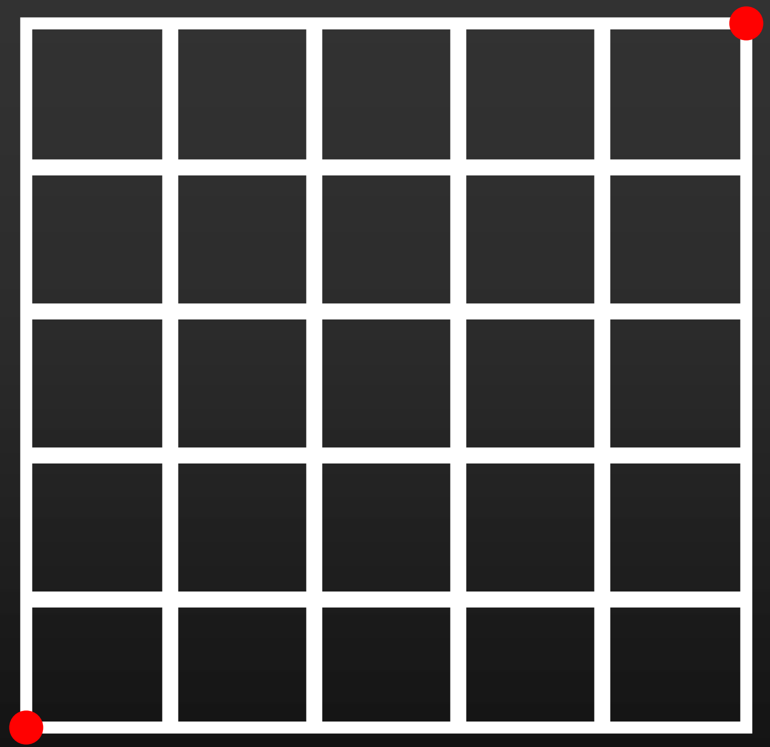


Background: Distance and Geodesic Computation via Dijkstra



- **Complexity:** O(n log n)
- **Advantage:** Works in any dimension and with any geometry (graphs)
- **Problems:**
  - Not consistent
  - Unorganized points?
  - Noise?
  - Implicit surfaces?

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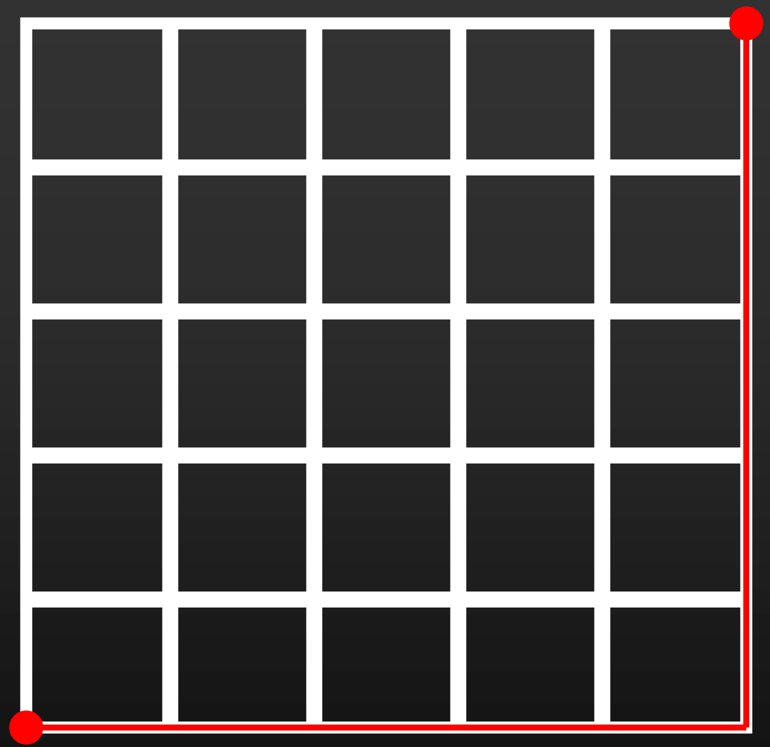

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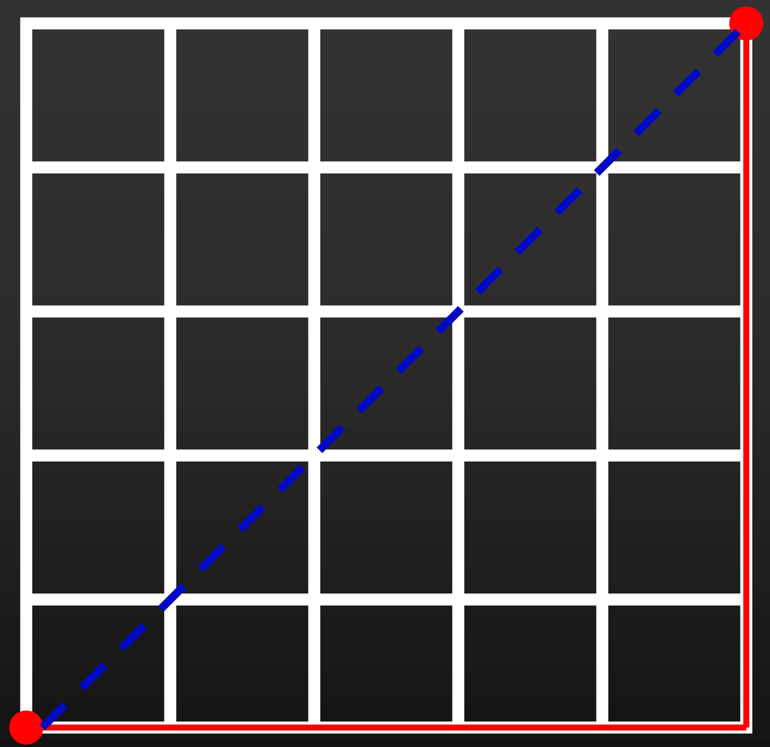

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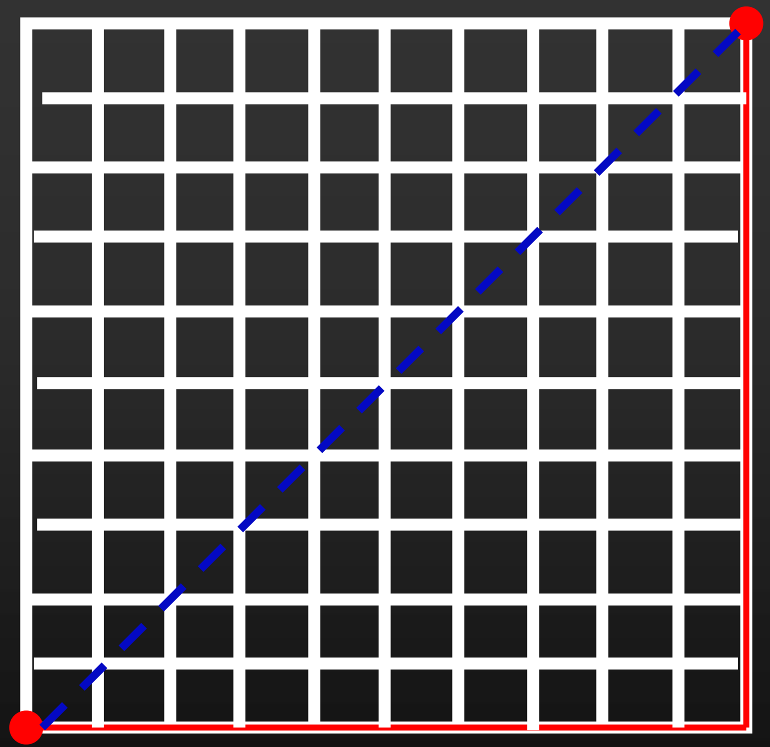

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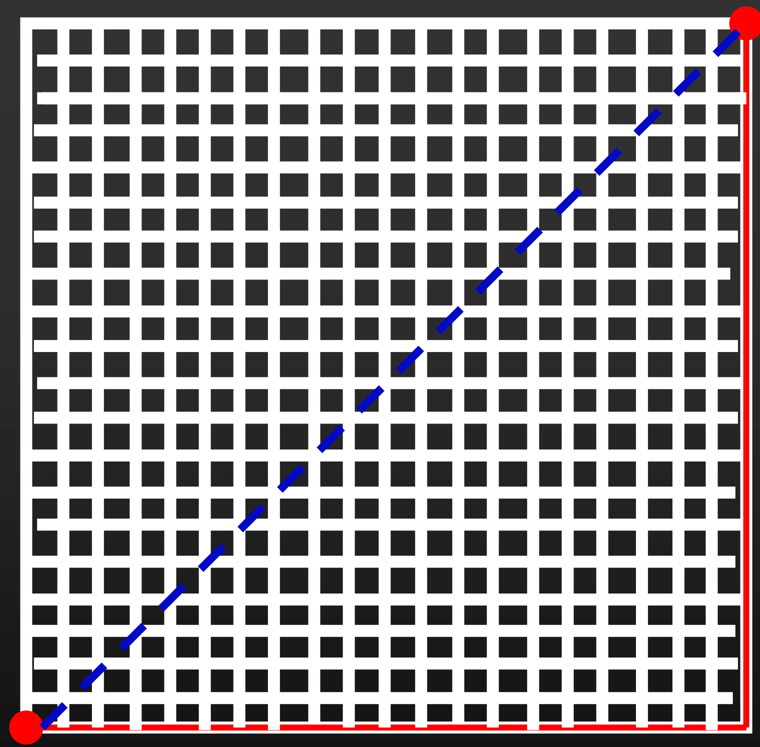

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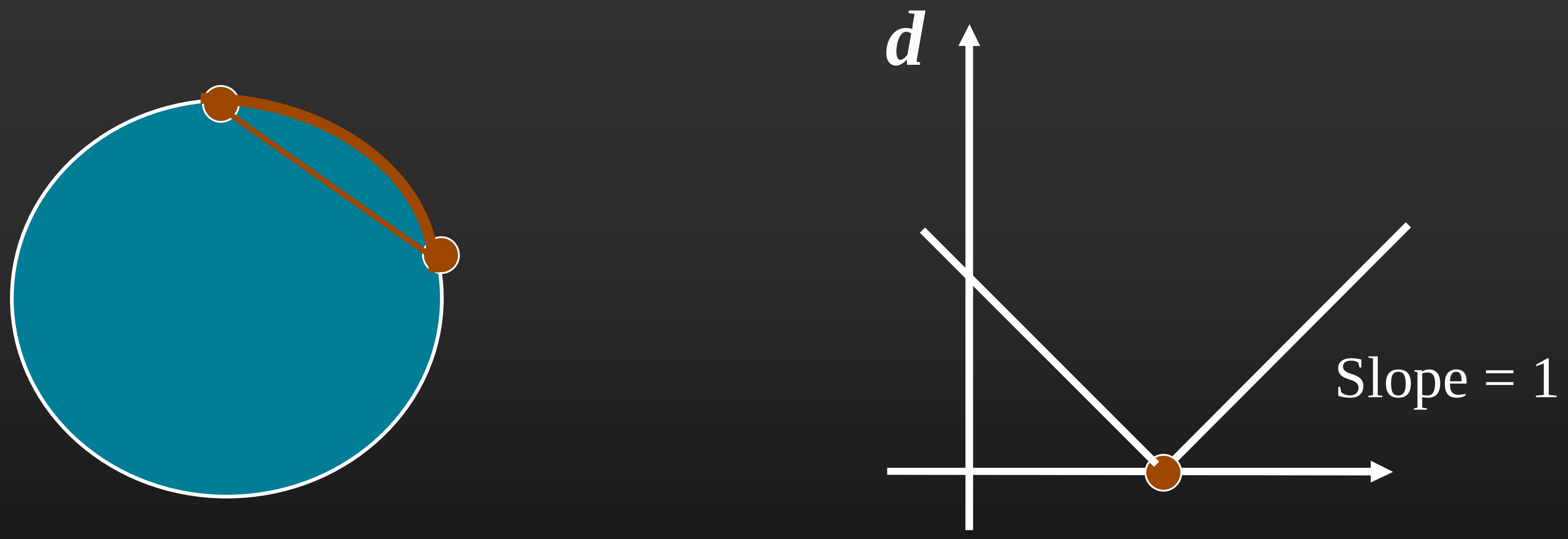
Background: Distance Functions as Hamilton-Jacobi Equations

- *g* = weight on the hyper-surface
- The *g*-weighted distance function between two points *p* and *x* on the hyper-surface *S* is:

$$\left\| \nabla_S d_S^g(p, x) \right\| = g$$

$$\left\| \nabla_S d_S^g(p,x) \right\| = g$$

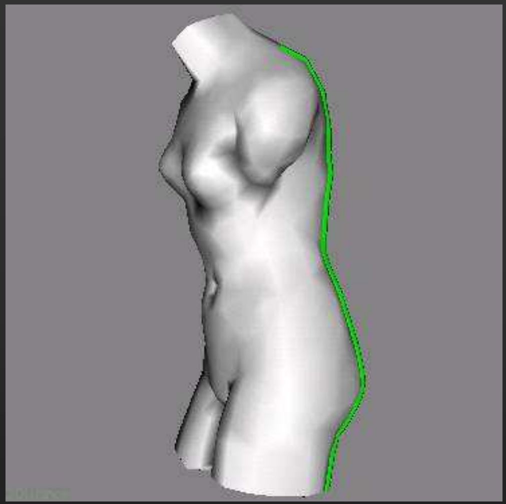


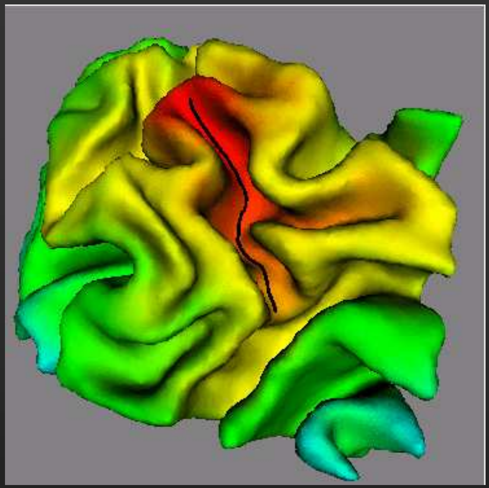
Background: Computing Distance Functions as Hamilton-Jacobi Equations

- Solved in O(n log n) by Tsitsiklis, by Sethian, and by Helmsen, **only** for Euclidean spaces and Cartesian grids

$$\left\| \nabla d^{g}(p, x) \right\| = g$$

- Solved **only** for acute 3D triangulations by Kimmel and Sethian





A real time [example]

The Problem

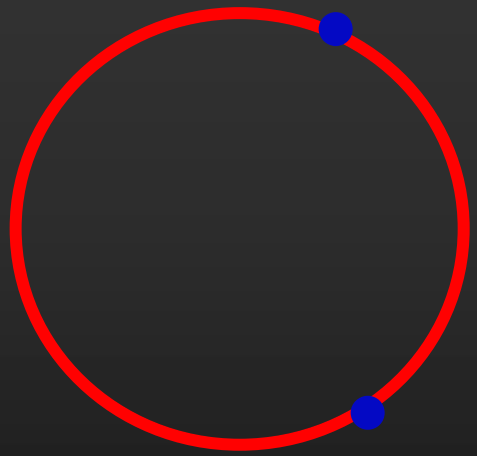
- **How to compute intrinsic distances and geodesics for**
  - General dimensions
  - Implicit surfaces
  - Unorganized noisy points (hyper-surfaces just given by examples)

Our Approach

- We have to solve

$$\left\| \nabla_S d_S^g(p, x) \right\| = g$$

Basic Idea



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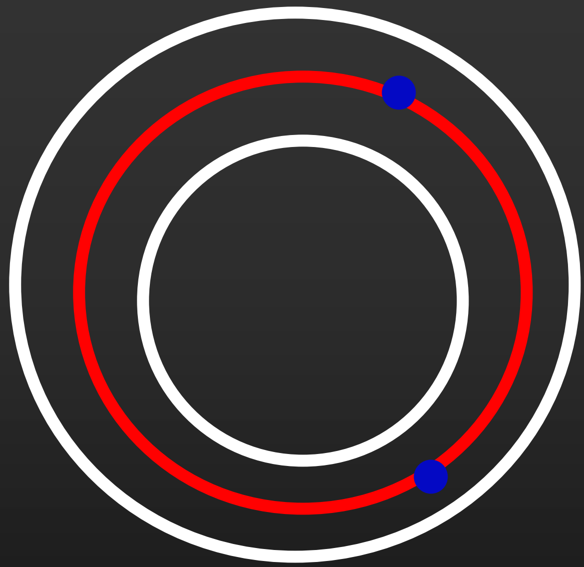

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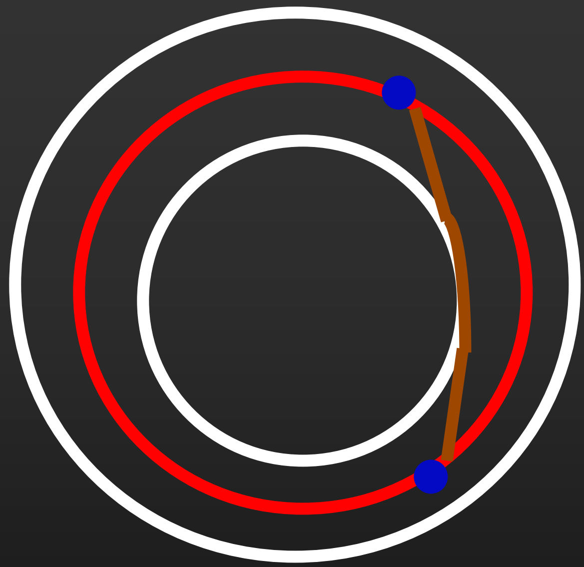
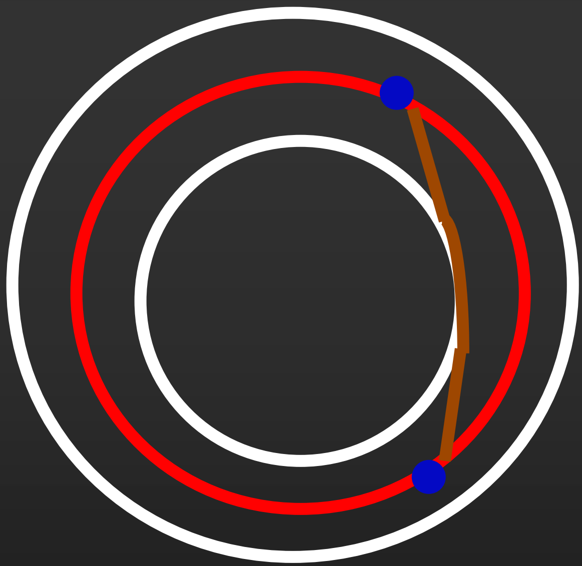


Theorem (Memoli-Sapiro):

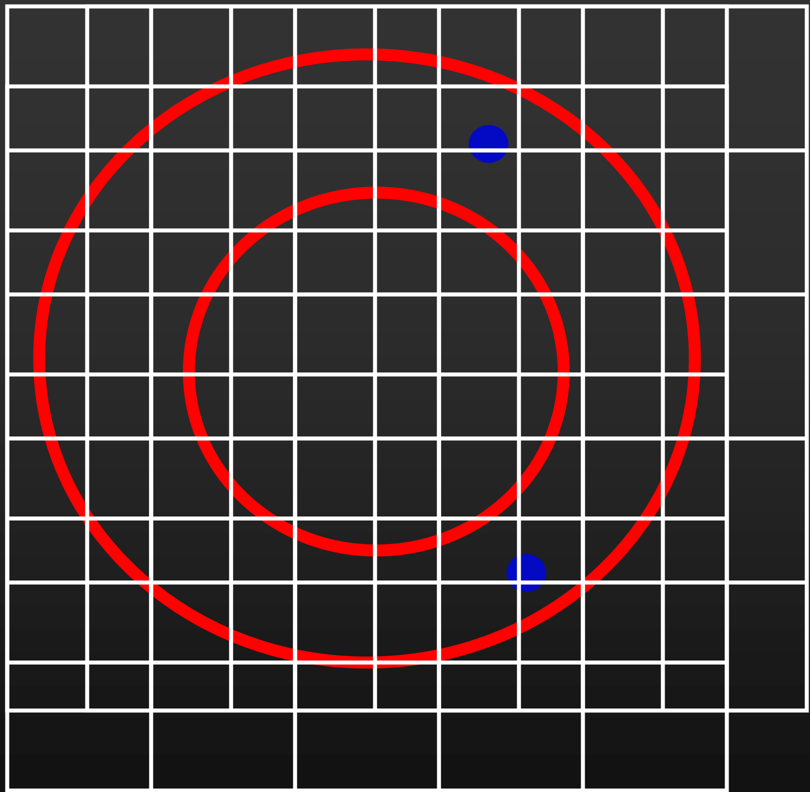
$$\left| d^{g} - d_{S}^{g} \right| \to 0$$

Basic idea



$$\left| d^g - d_S^g \right| \to \begin{cases} h^{1/2} & \text{general} \\ h & \text{local analytic} \\ h^\gamma, \gamma > 1 & \text{"smart" metric} \end{cases}$$

Why is this good?



$$\left\| \nabla_S d_S^g(p,x) \right\| = g$$

$$\Downarrow$$

$$\left\| \nabla d^g(p,x) \right\| = g$$

Implicit Form Representation

$$\text{S} = \text{level} - \text{set of } \Psi : R^n \rightarrow R = \{x : \Psi(x) = 0\}$$

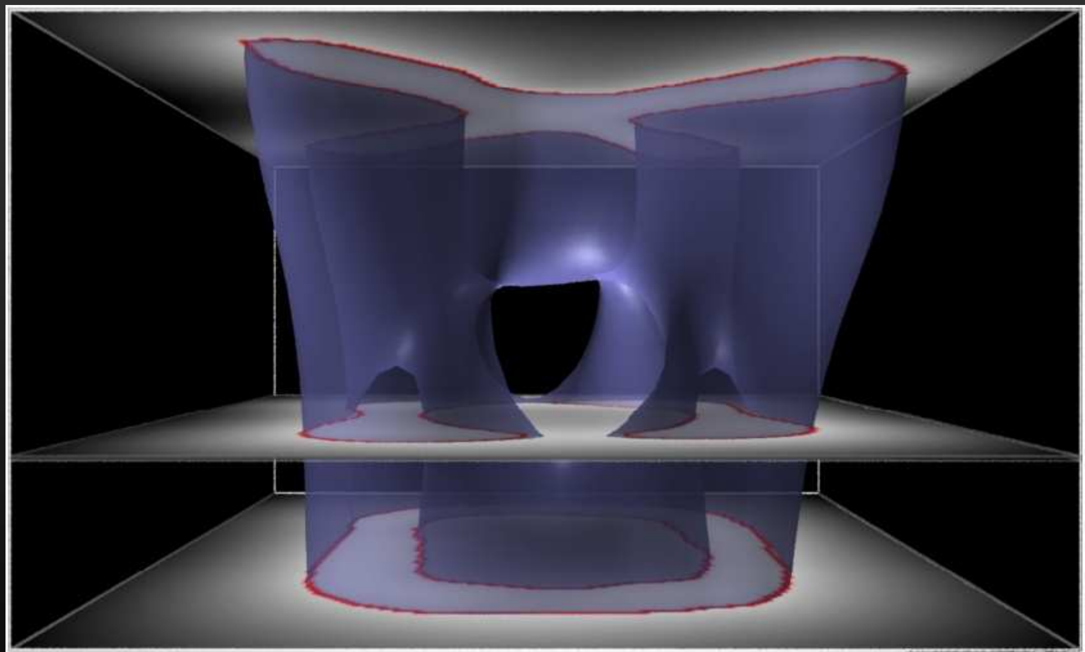


Figure from G. Turk

Data extension

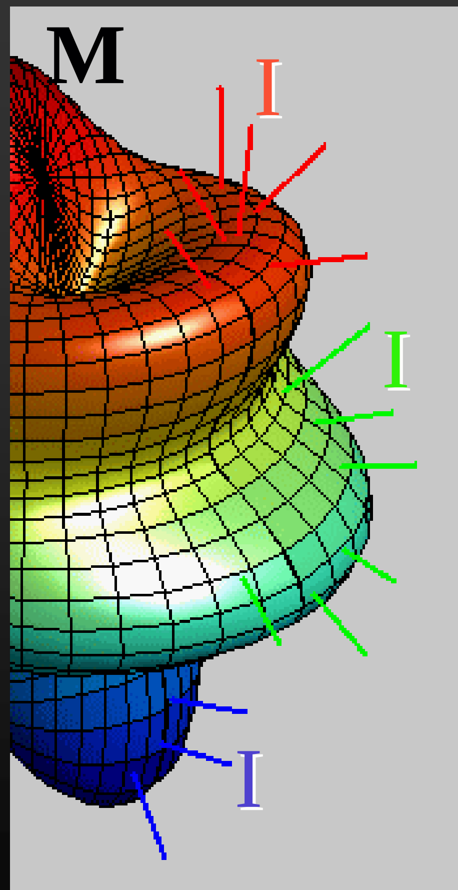
$$I : M \to R$$

- Embed M:

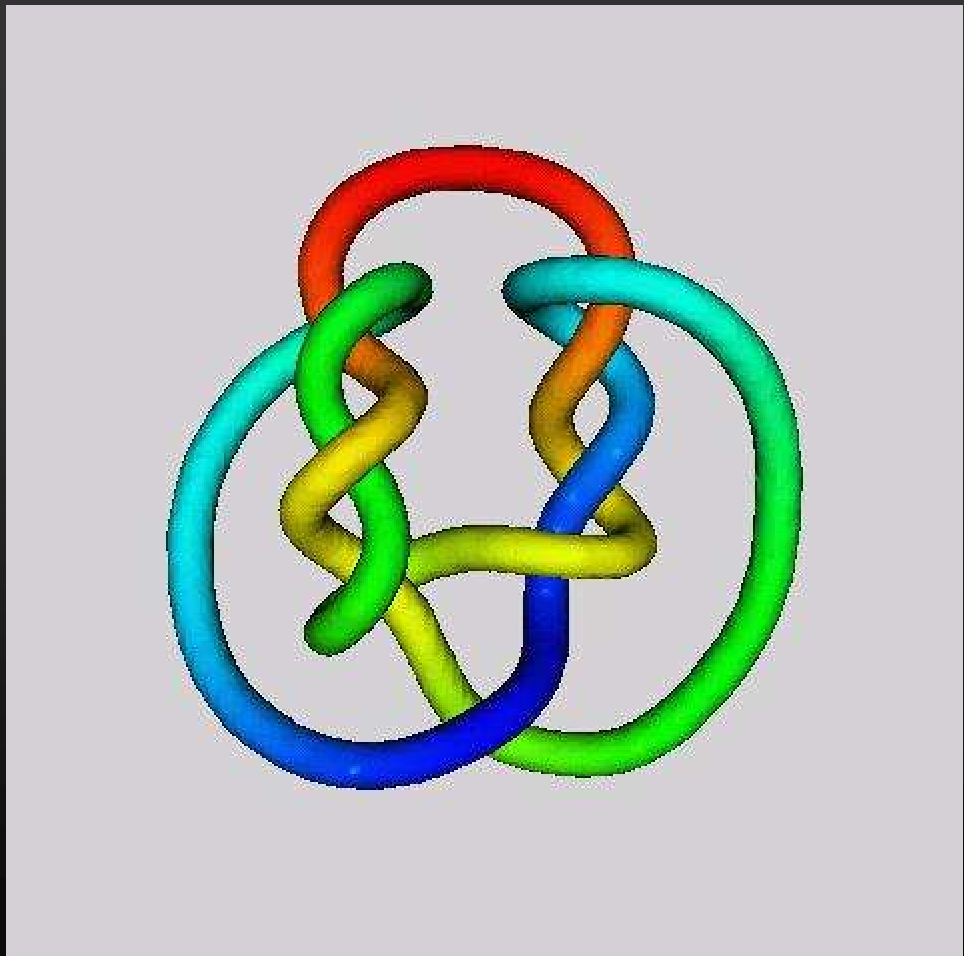
$$M = \{x : \Psi(x) = 0\}$$

- Extend I outside M:

$$\frac{\partial I}{\partial t} + sign(\psi)\,(\nabla I \cdot \nabla \psi) = 0$$



Examples



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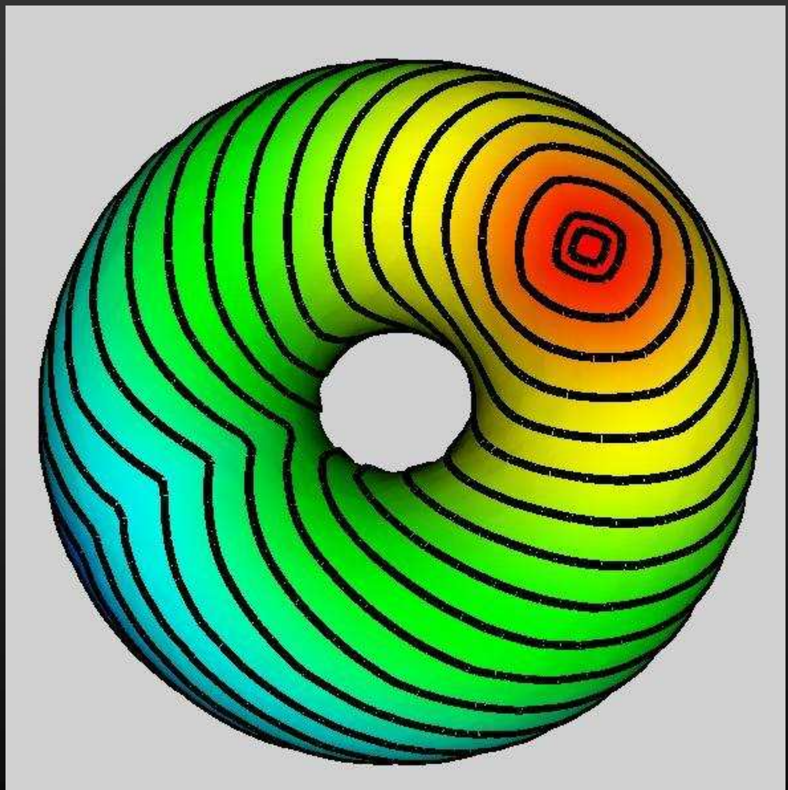

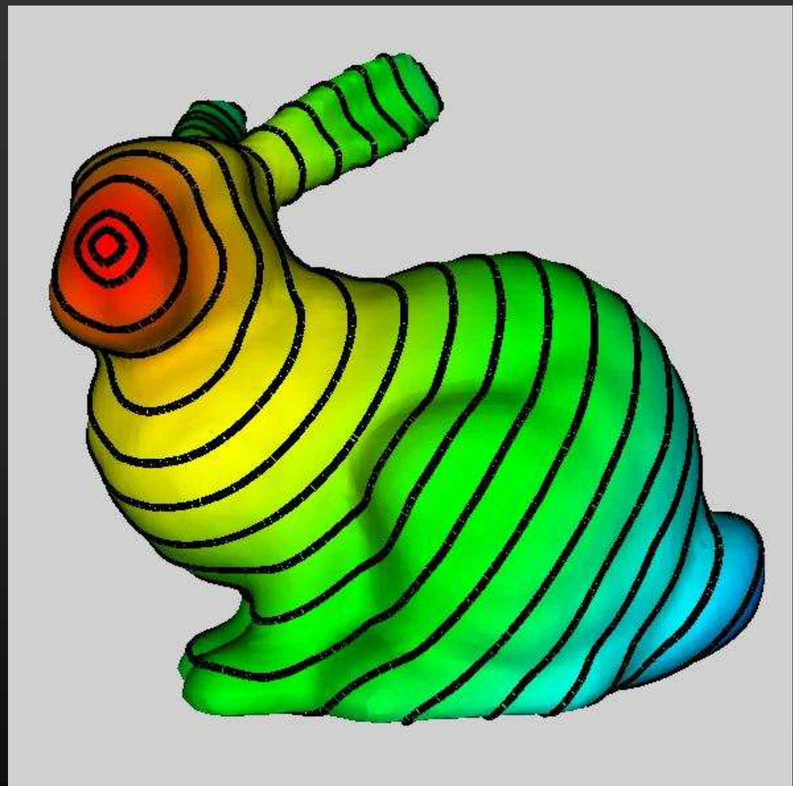

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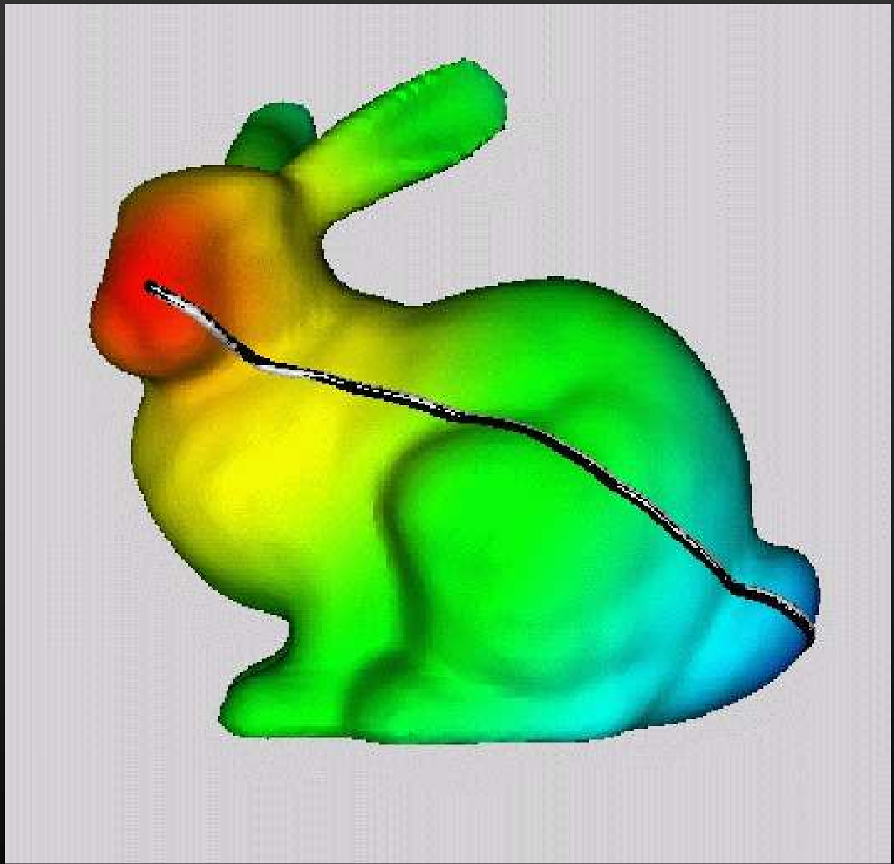

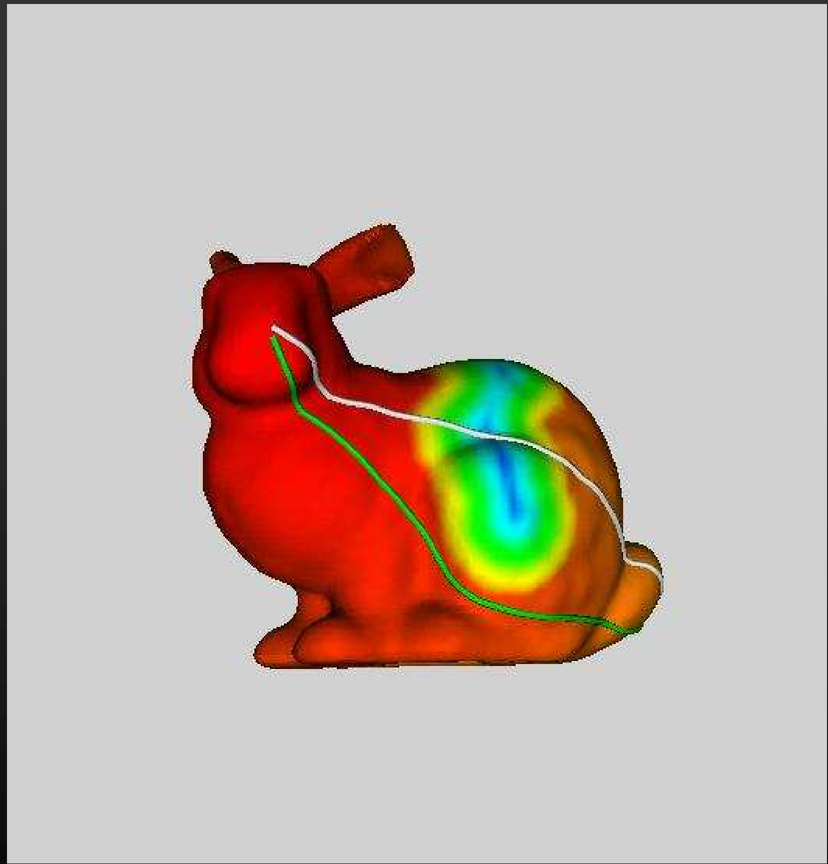

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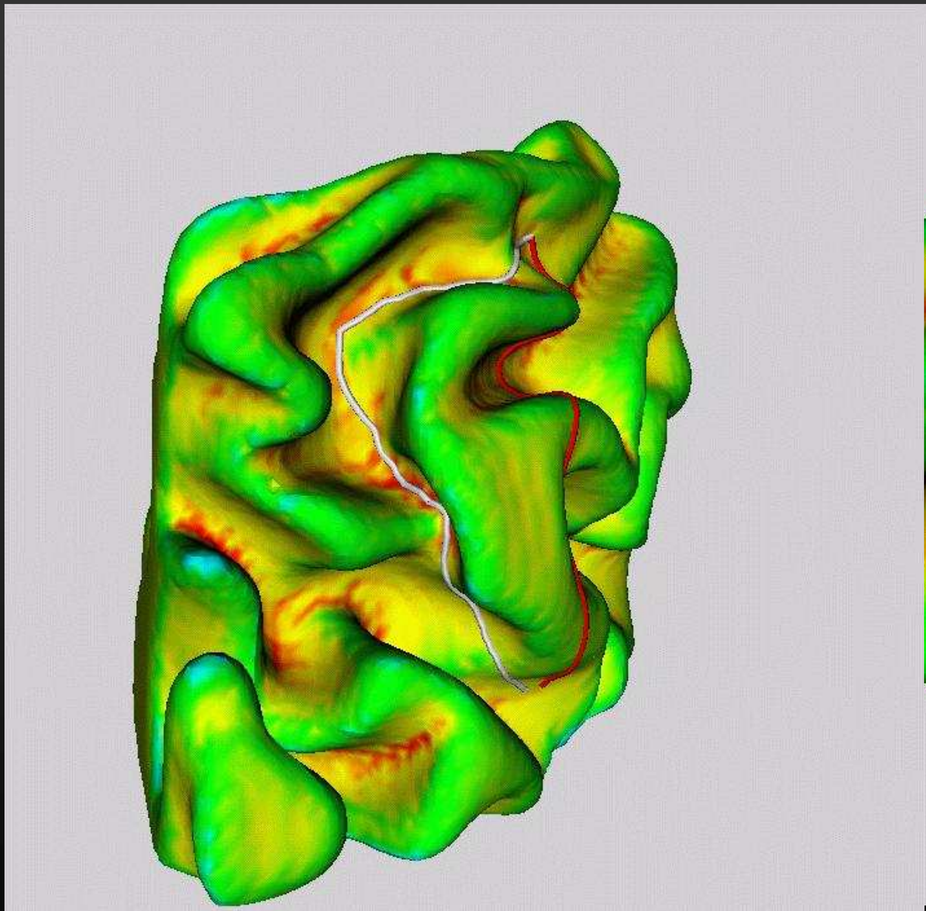

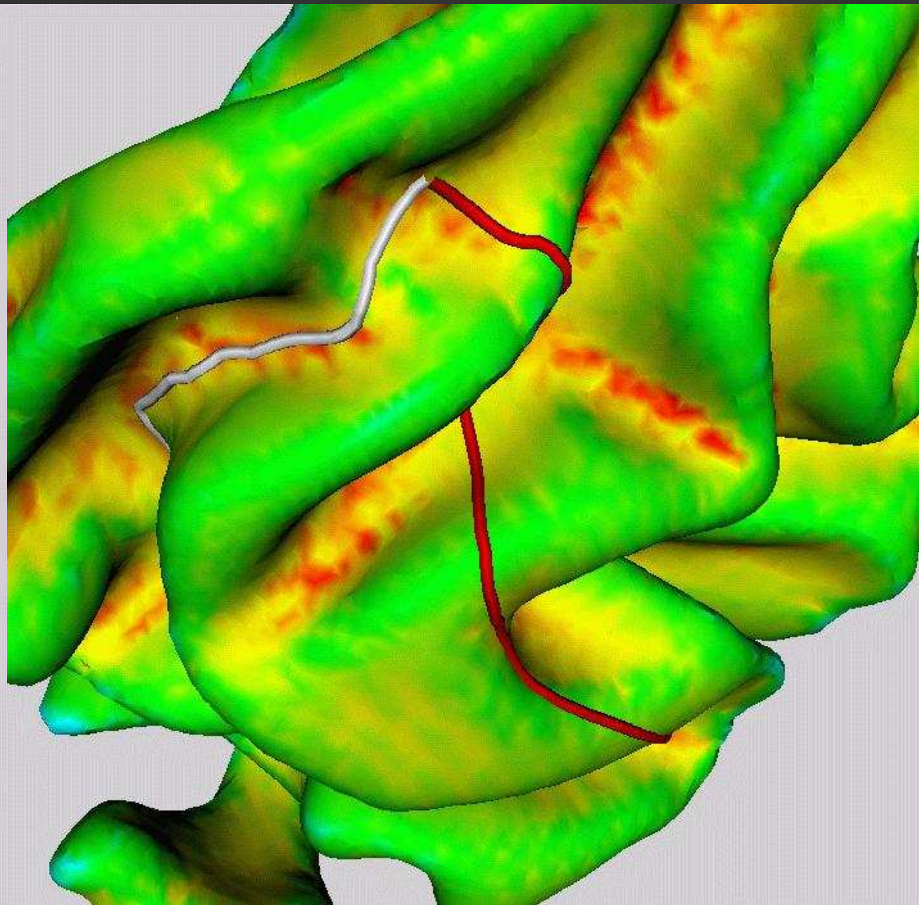

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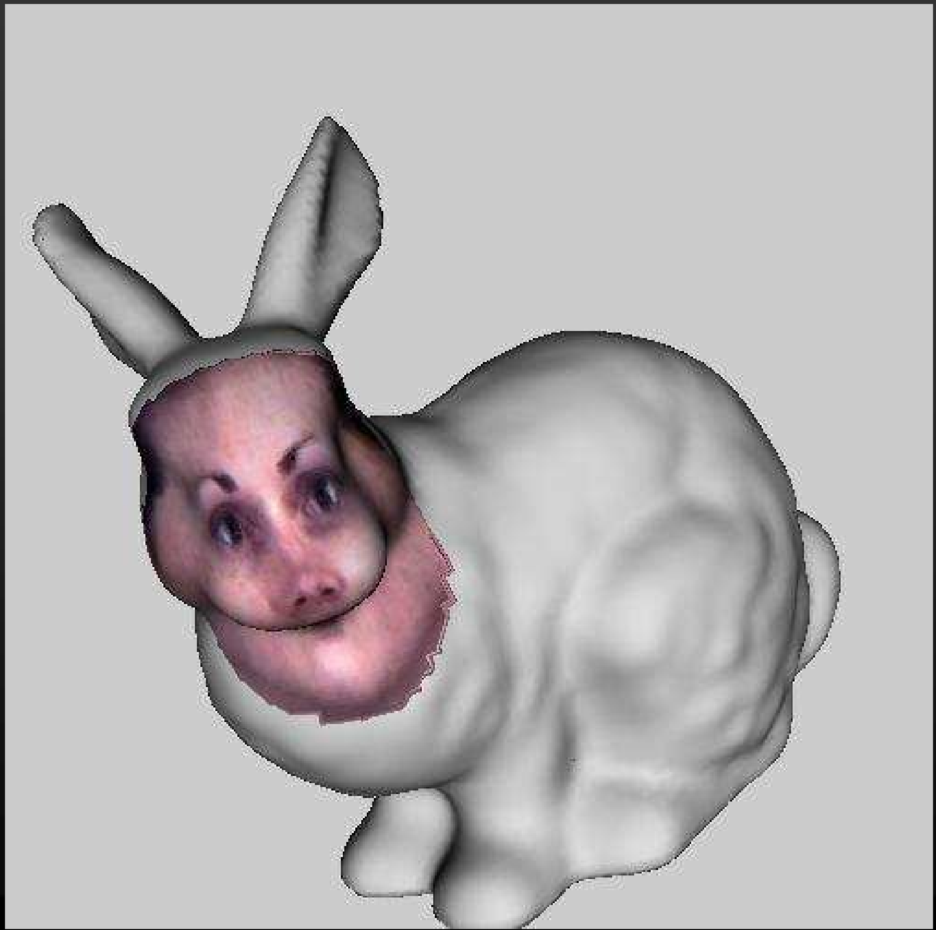
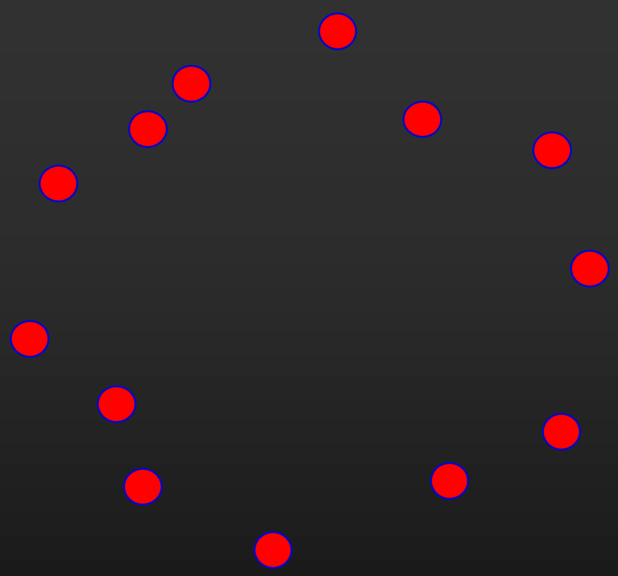
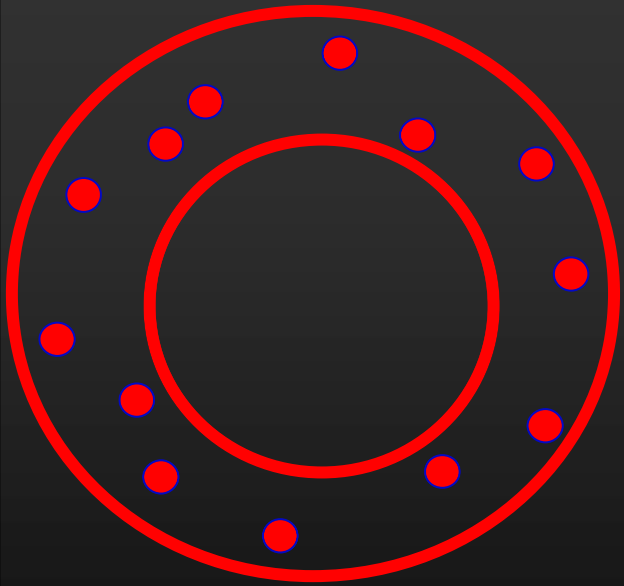
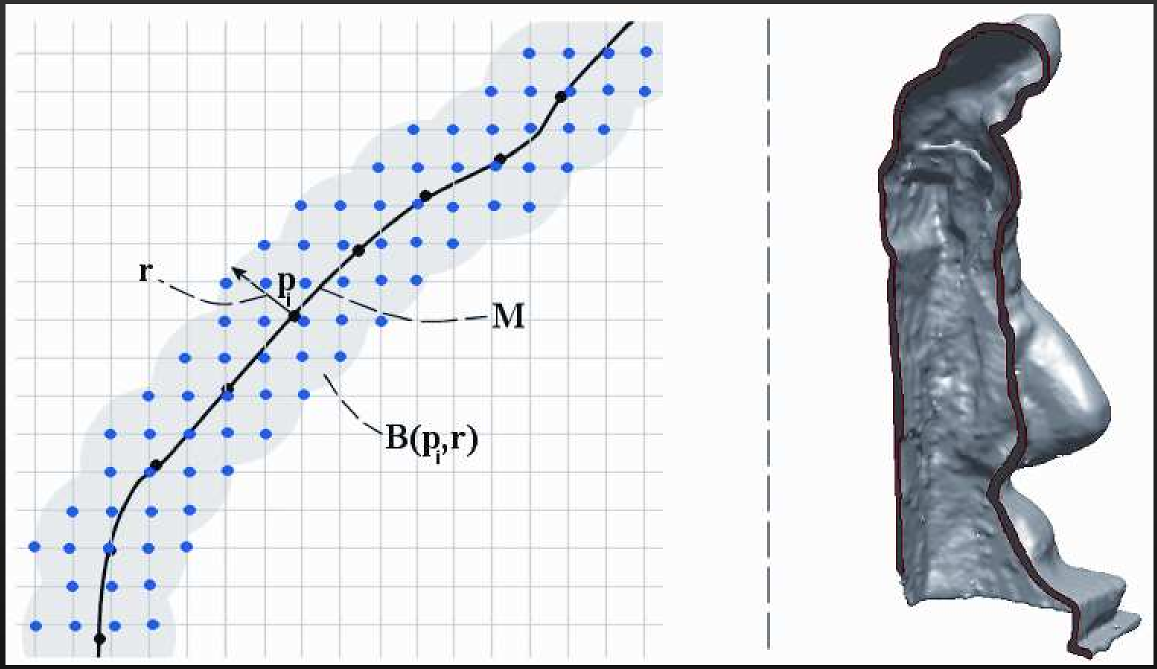


Unorganized points



Unorganized points (cont.)





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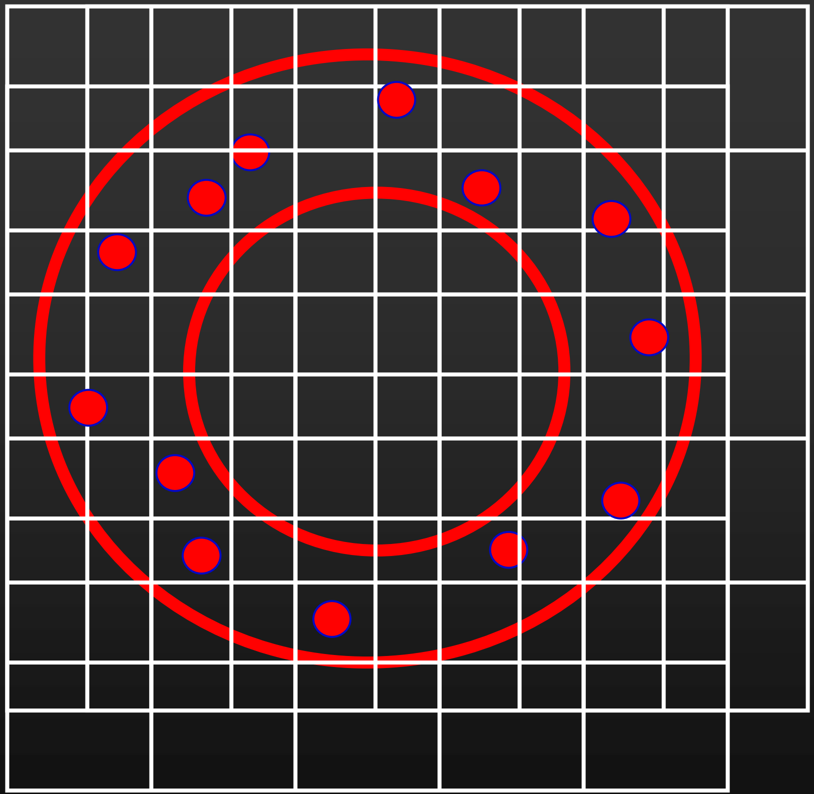
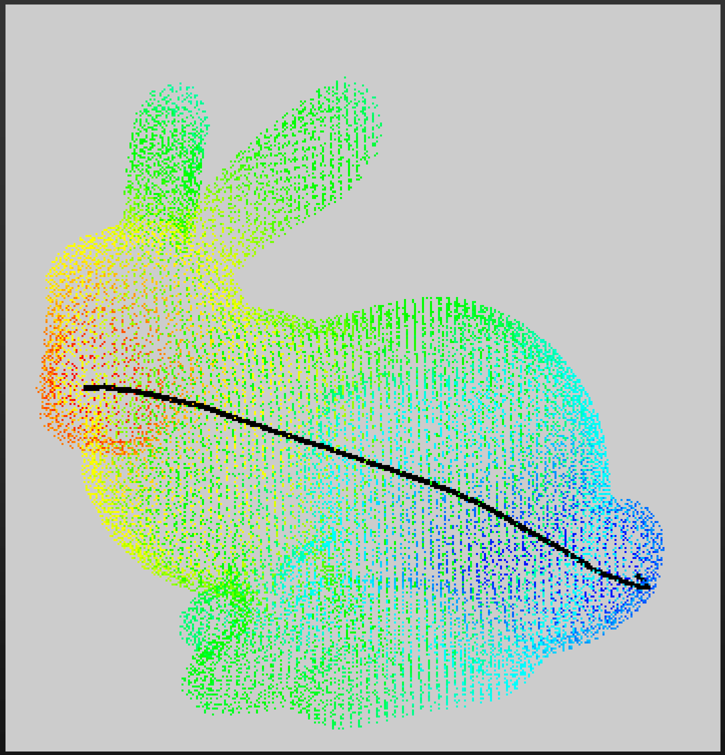




Randomly sampled manifolds (with noise)

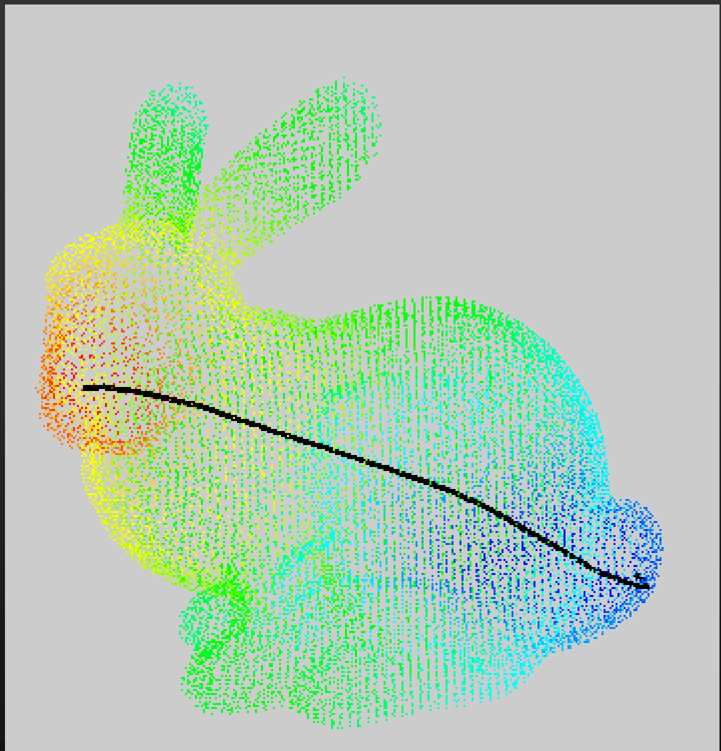
Theorem (Memoli - S. 2002):

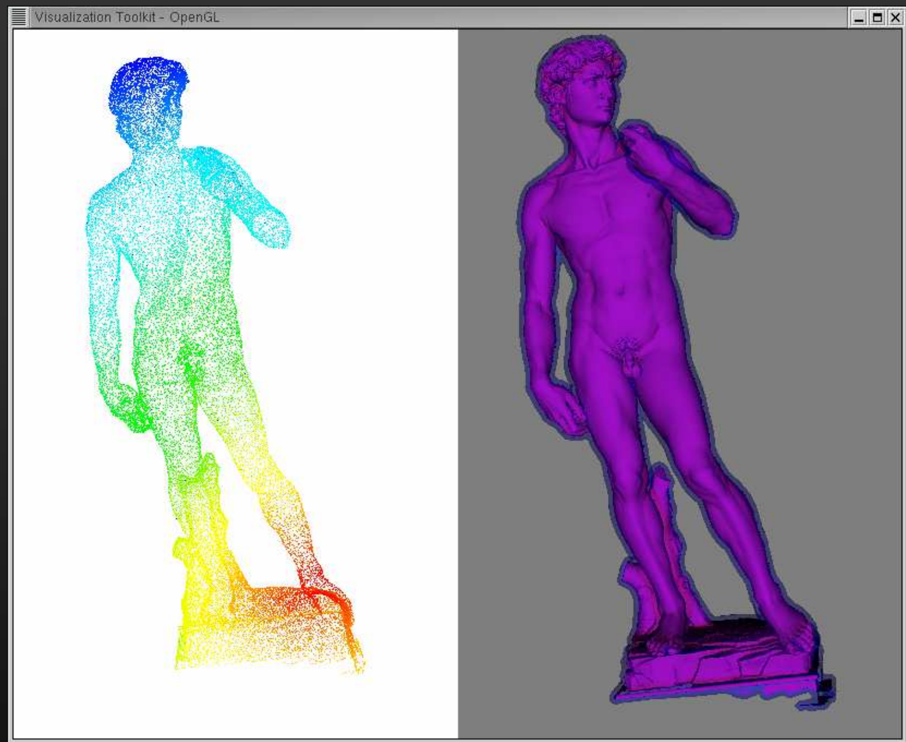
$$\max_{p,q\in\mathcal{S}}\left(d_{\mathcal{S}}(p,q)-d_{\Omega^h_{\mathcal{P}_{n(h)}(h)}}(p,q)\right)\leq C_{\mathcal{S}}\sqrt{h}$$

$$P(\max_{p,q\in calS}\left(d_{\mathcal{S}}(p,q)-d_{\Omega^h_{\mathcal{P}_n}}(p,q)\right)>\varepsilon)\xrightarrow{n\uparrow\infty}0$$

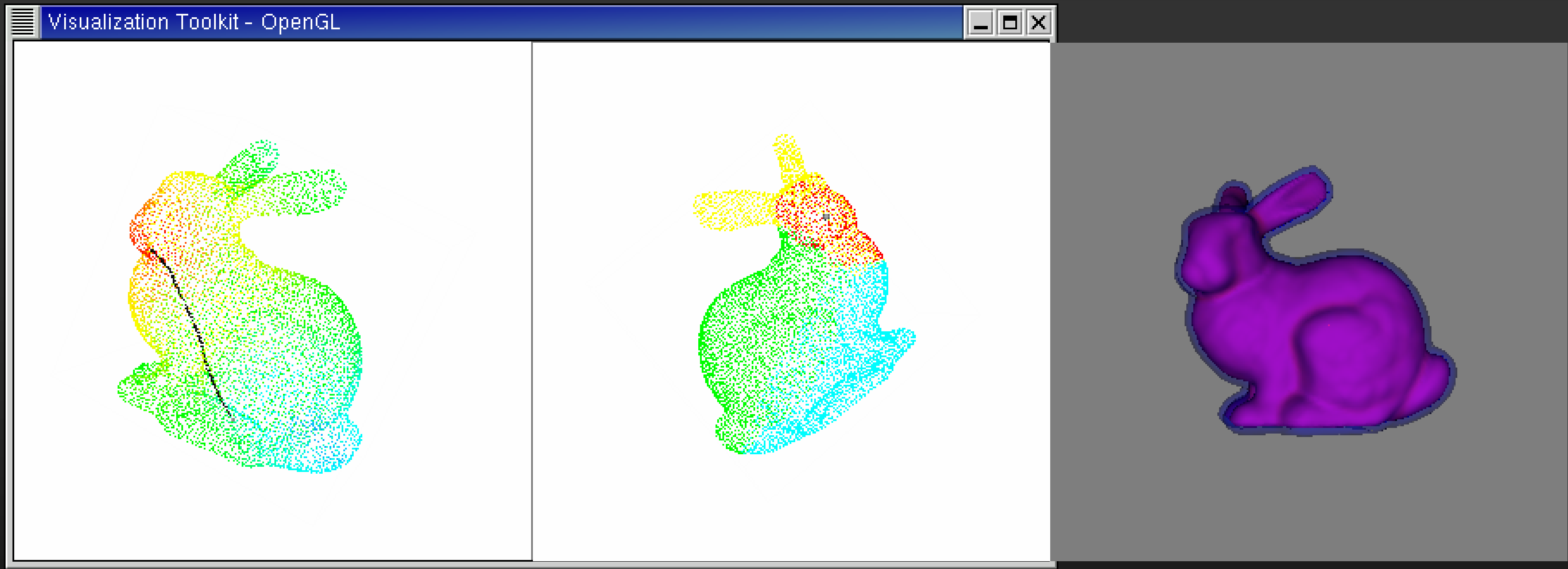
$$\lim_{h,n}P(d_{\mathcal{H}}(\mathcal{S},\Omega^h_{\mathcal{P}_n})>\varepsilon)=0$$

Examples (VRML)

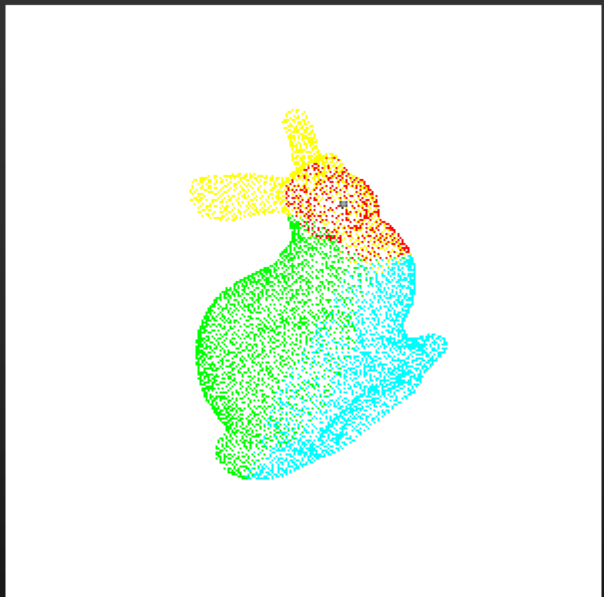




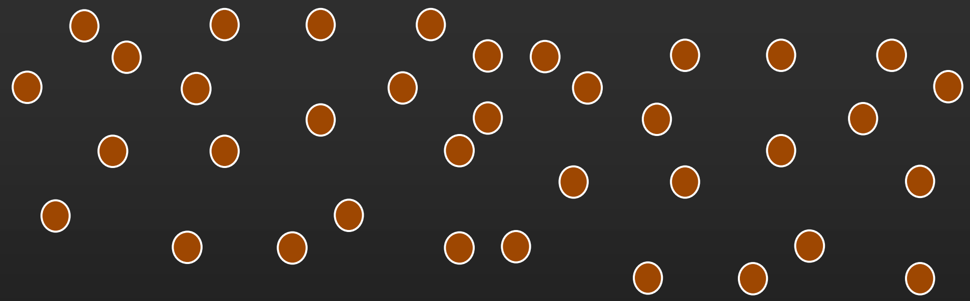
Examples

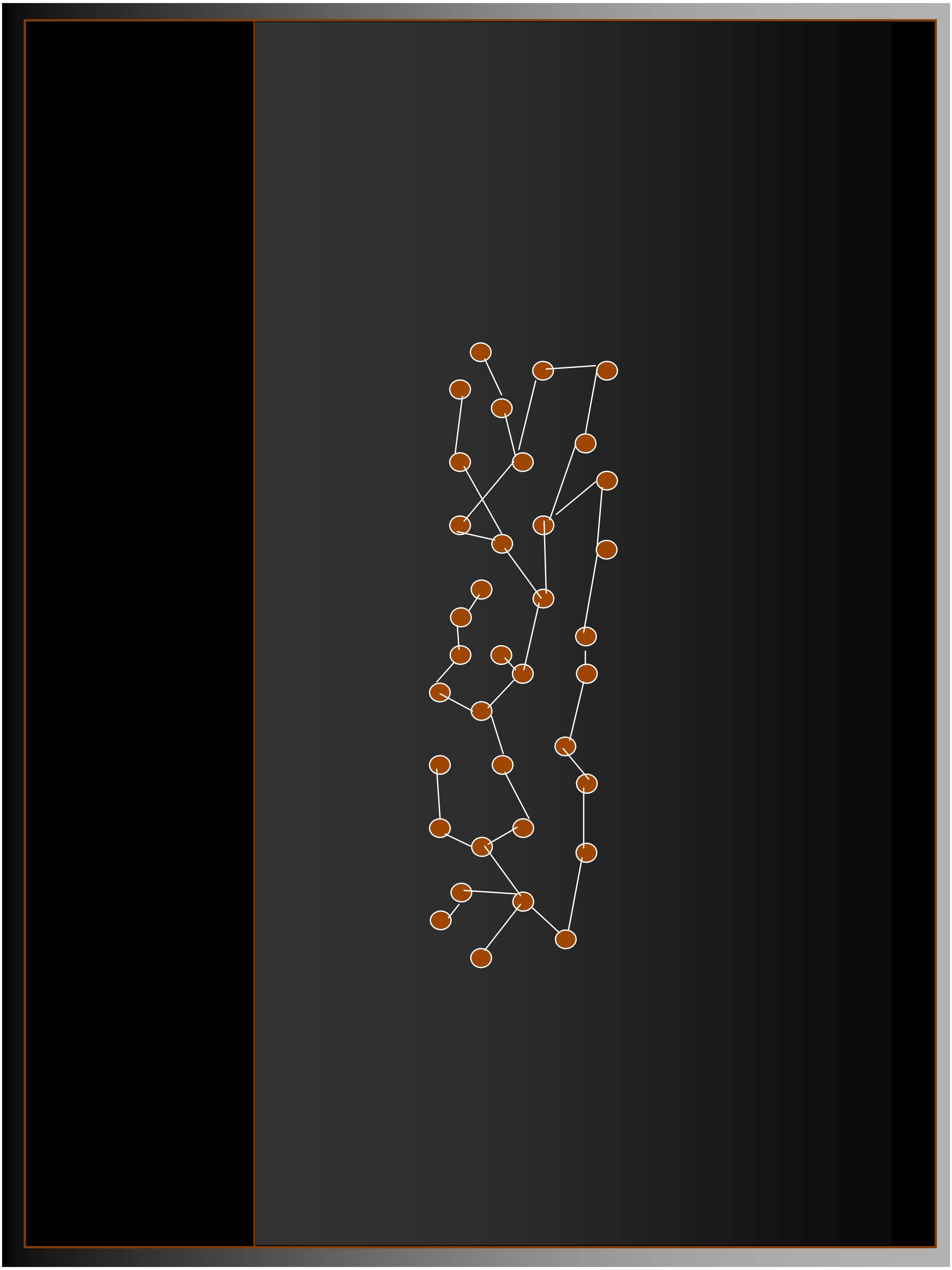


Intrinsic Voronoi of Point Clouds



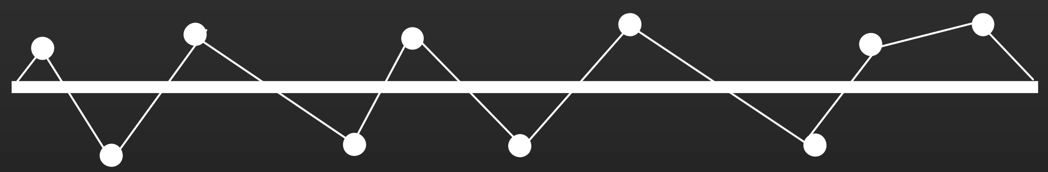
Intermezzo:
de Silva, Tenenbaum, et al...

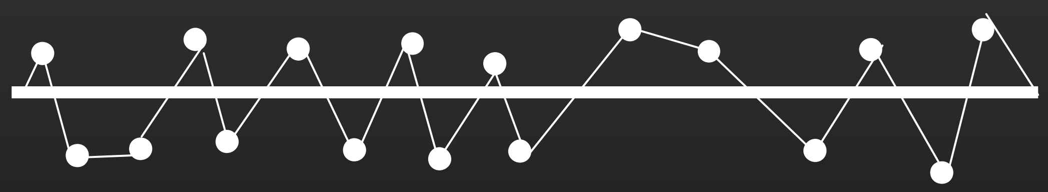




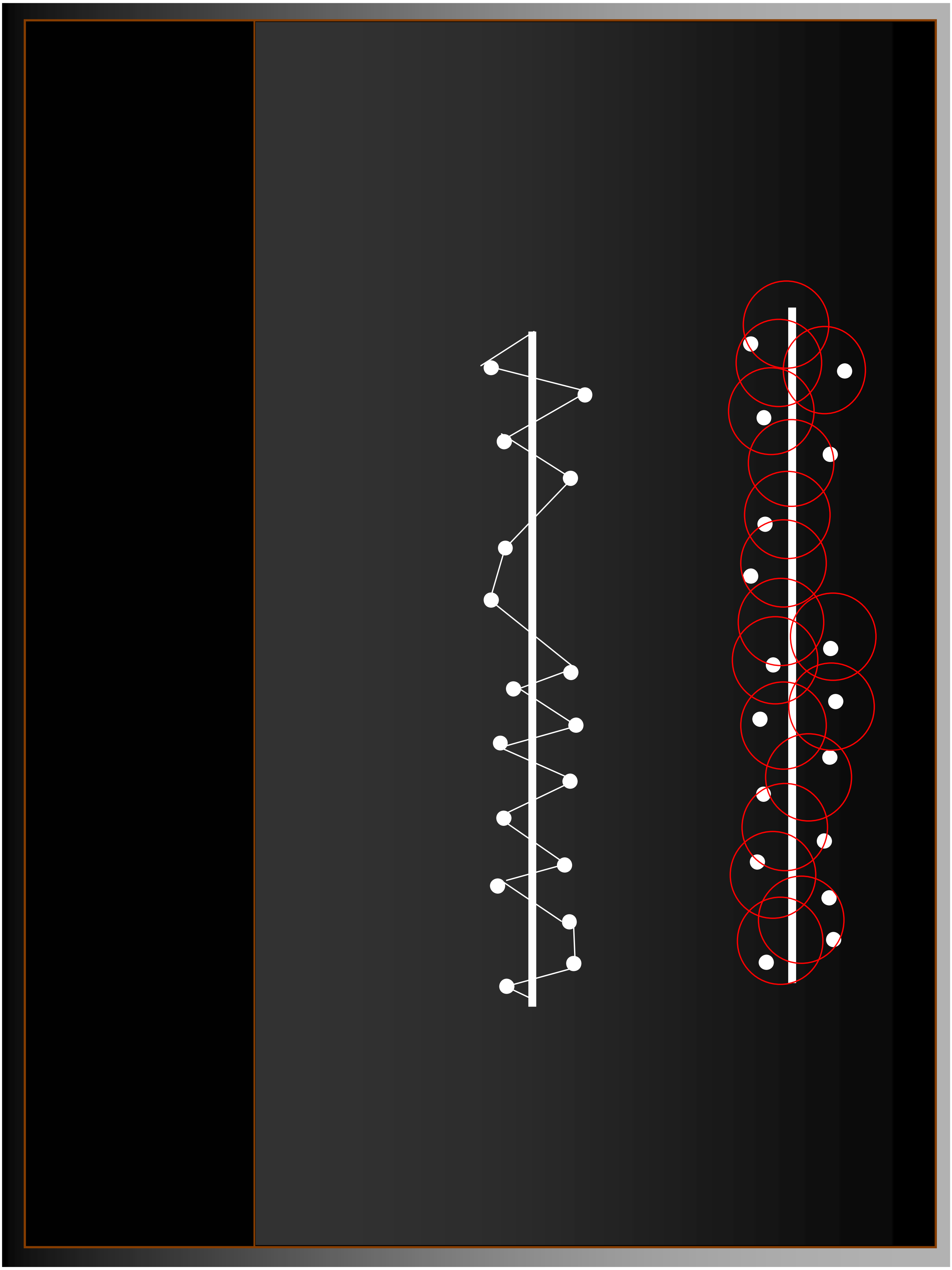
Intermezzo:
Tenenbaum, de Silva, et al...

- **Main Problem:**
  - Doesn’t address noisy examples/measurements:
    Much less robust to noise!





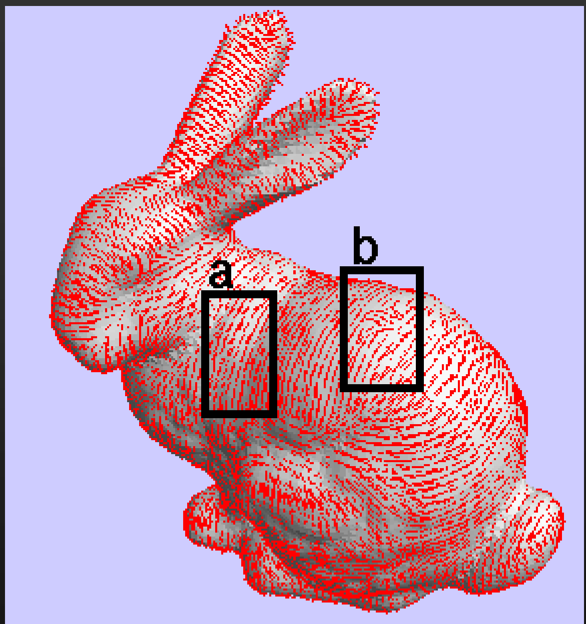
Error increases with the number of samples!



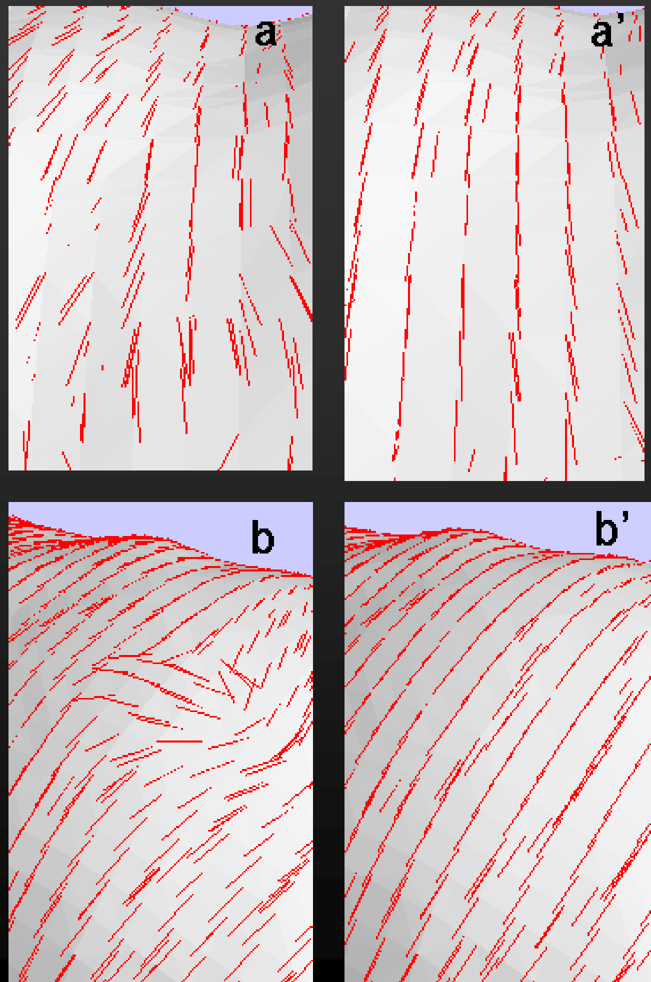
Intermezzo:
de Silva, Tenenbaum, et al...

- **Problems:**
  - Doesn’t address noisy examples/measurements: Much less robust to noise!
  - Only convex surfaces
  - Uses Dijkestra (back to non consistency)
  - Doesn’t work for implicit surface representations

Is this a geodesic?



noisy



cleaned

Generalized geodesics: Harmonic maps

- Find a smooth map from two manifolds *(M,g)* and *(N,h)* such that

$$\min_{C:M\to N} \int_{\Omega} \left\| \nabla_M C \right\|^p \, dvol_M$$

$$\left(\frac{\partial C}{\partial t} = \right) \qquad \Delta_M C + A_N(C) < \nabla_M C, \nabla_M C >= 0$$

Examples

- M is an Euclidean space and N the real line

$$\Delta C = 0$$

- M = [0,1], geodesics!

$$\frac{\partial^2 C}{\partial t^2} + A_N(C) < \nabla_M C, \nabla_M C >= 0$$

Color Image Enhancement

(with B. Tang and V. Caselles)

Implicit surfaces

- Domain and target are implicitly represented: Simple Cartesian numerics

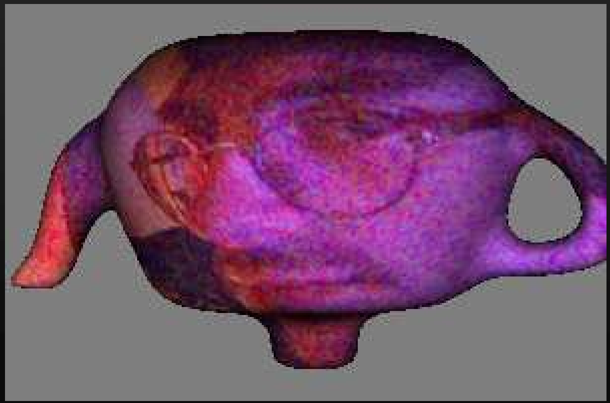
$$\frac{\partial C}{\partial t} = \mathrm{div}\left(P_{\nabla\Psi}\nabla C\right) + \left(\sum_k H_\Phi \left\langle \frac{\partial C}{\partial x_k}, \frac{\partial C}{\partial x_k} \right\rangle\right) \|\nabla\Phi\|$$

Example: Chroma denoising on a surface (with Bertalmio, Cheng, Osher)

original

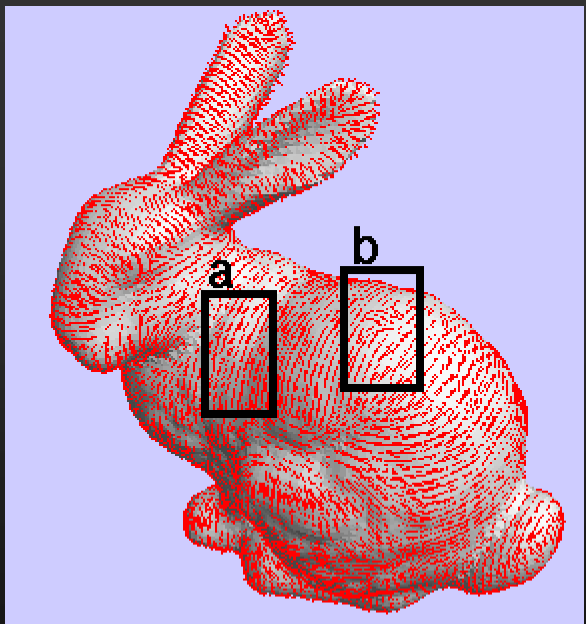
noisy

enhanced

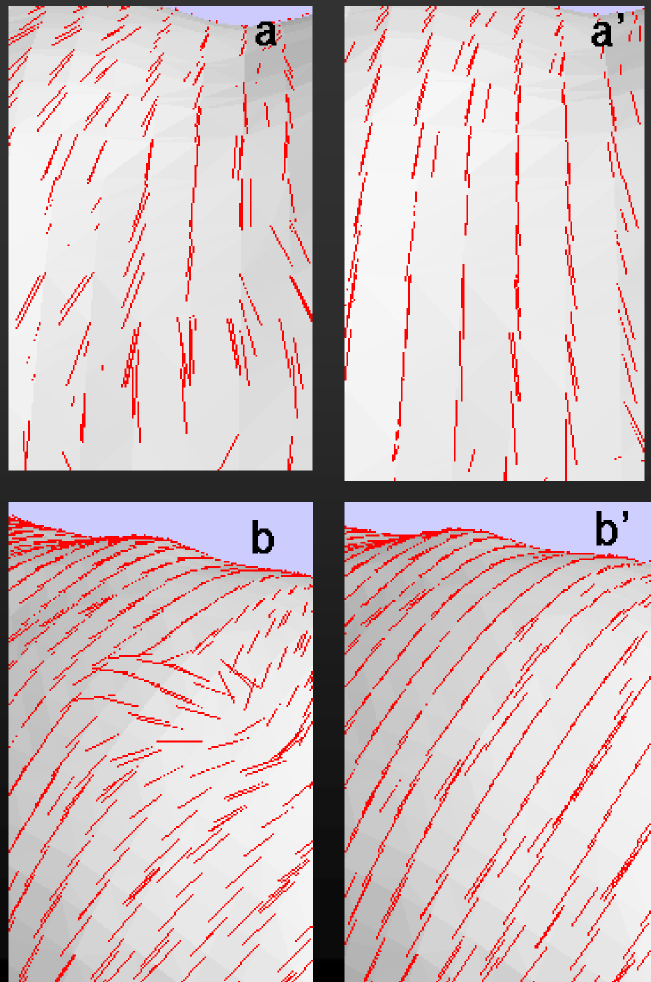


Example: Direction denoising

(with Bertalmio, Cheng, Osher)



noisy



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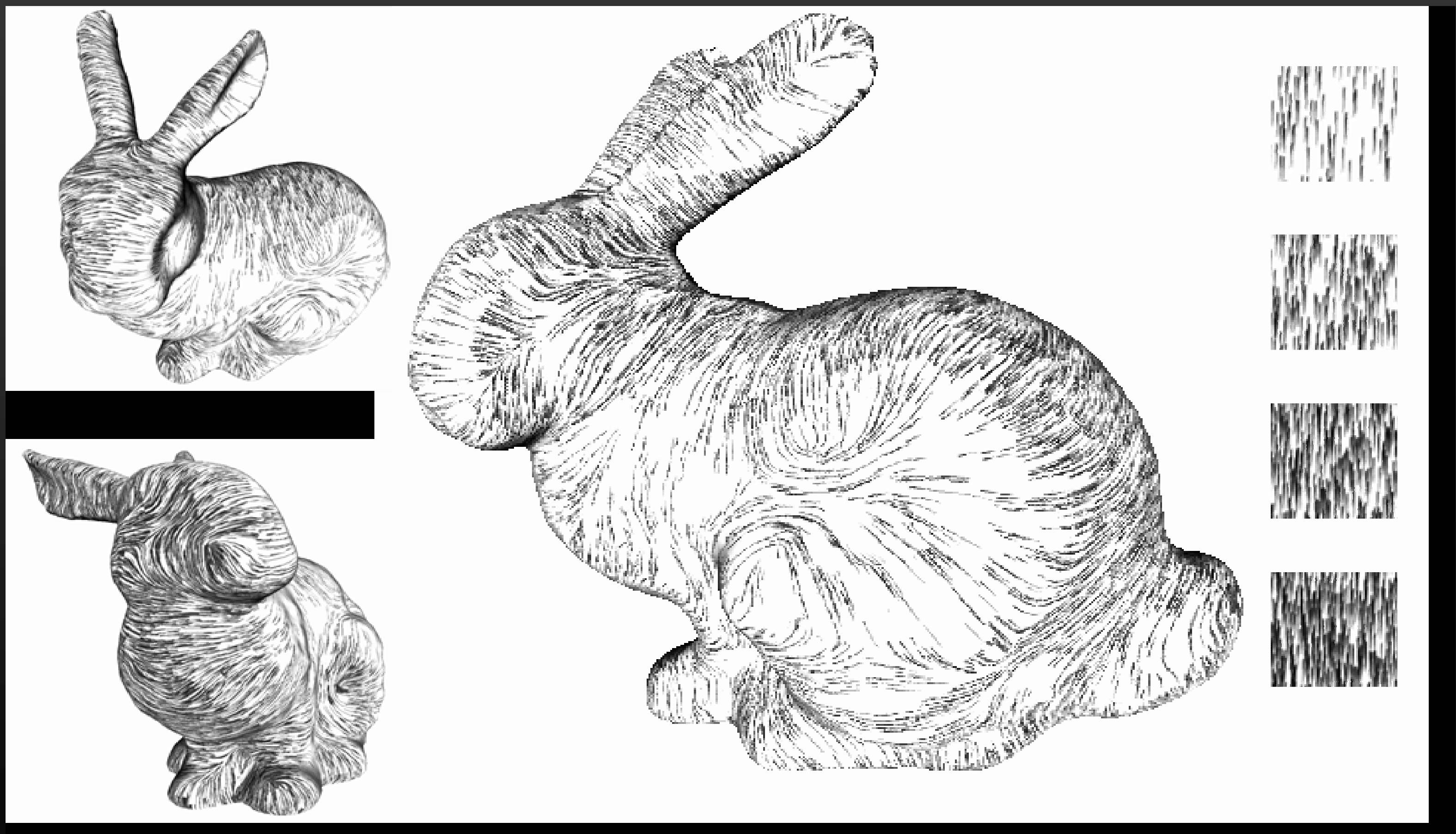
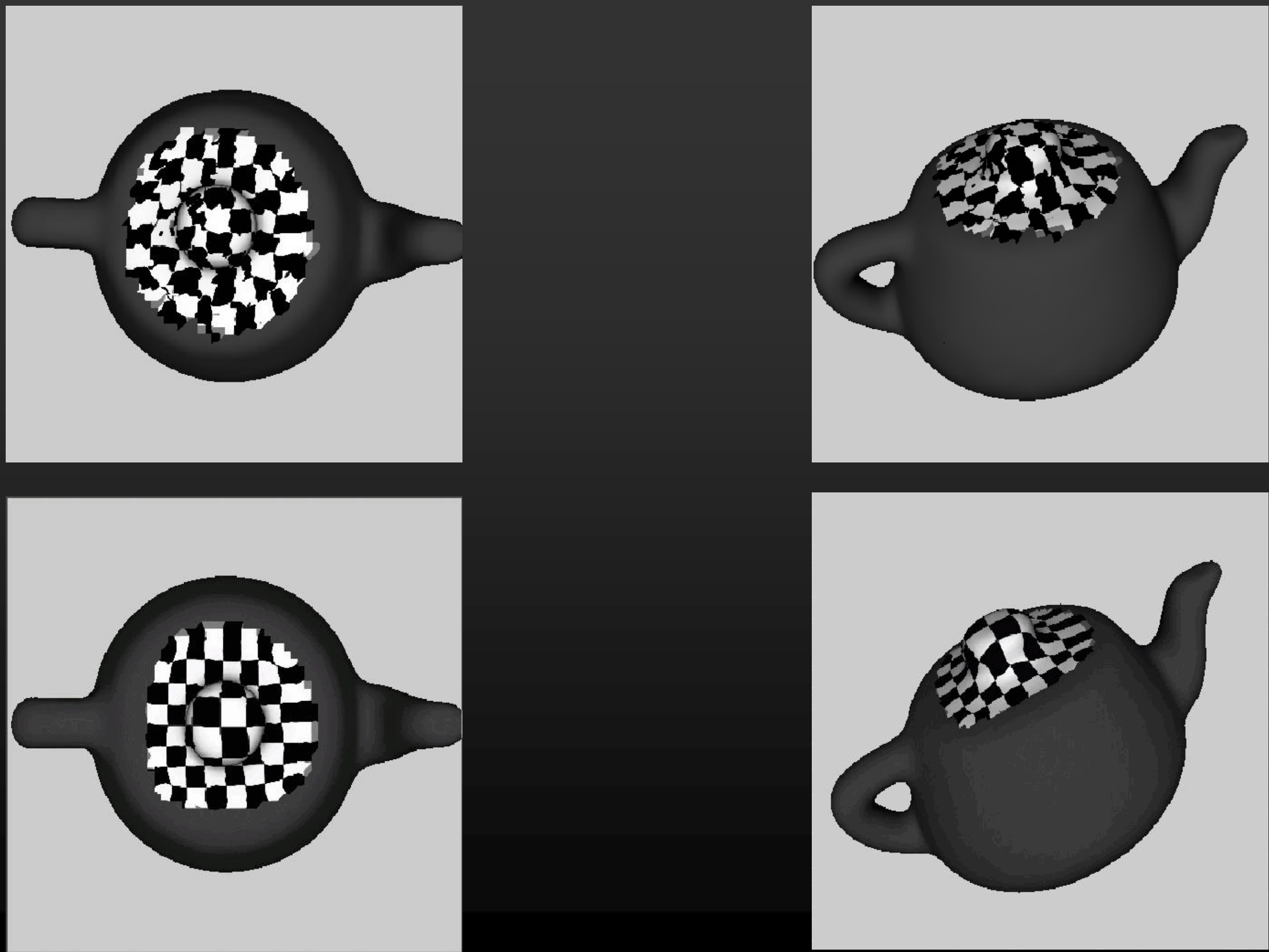
Application (with G. Gorla and V. Interrante)



Texture mapping denoising

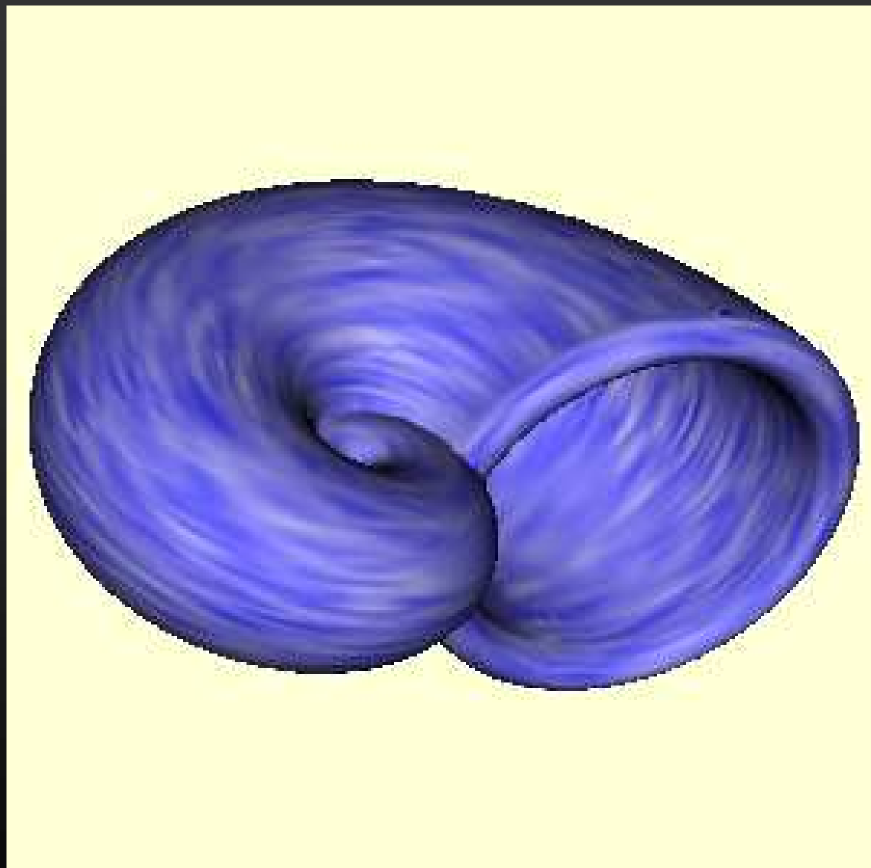


Texture mapping denoising

Examples
(with Betalmio, Cheng, Osher)

Vector field visualization (e.g., principal directions)

(with Bertalmio, Cheng, Osher)



Concluding remarks

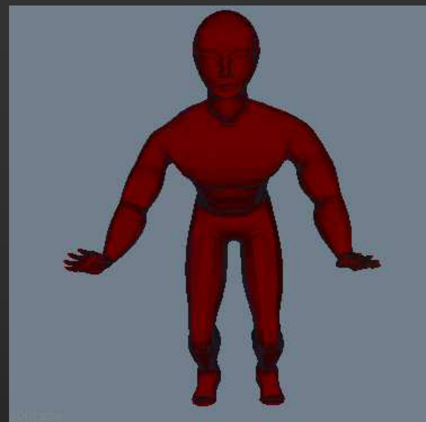
- A general computational framework for distance functions, geodesics, and generalized geodesics
- Implicit hyper-surfaces and un-organized points

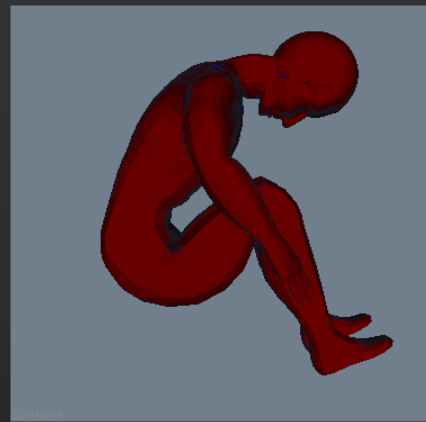
Comparing Point Clouds

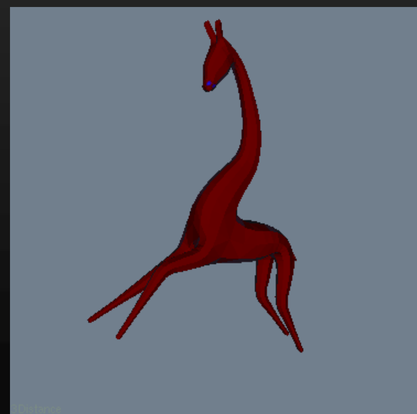
Joint with Facundo Memoli

What is and Motivation

- **Comparing point clouds**
  - Dimension independent
  - Geometric
  - Bending (isometric) invariant
  - Supported by theory and computational framework







The Gromov-Hausdorff Distance

- **Hausdorff distance**

$$d_{\mathcal{H}}^{Z}(X,Y) \triangleq \max\left(\sup_{x\in X} d(x,Y), \sup_{y\in Y} d(y,X)\right)$$

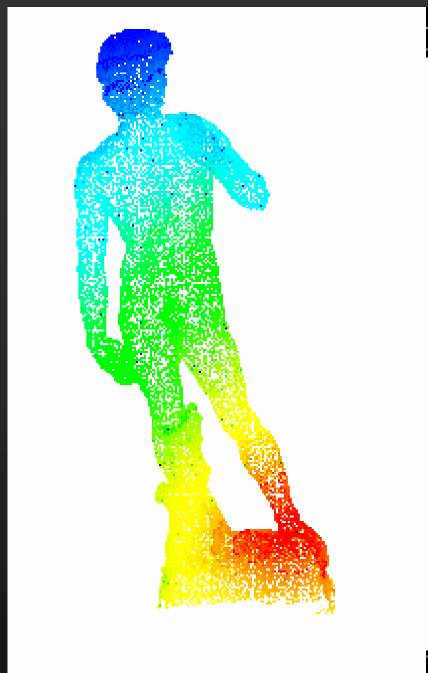
- **Gromov-Hausdorff distance**

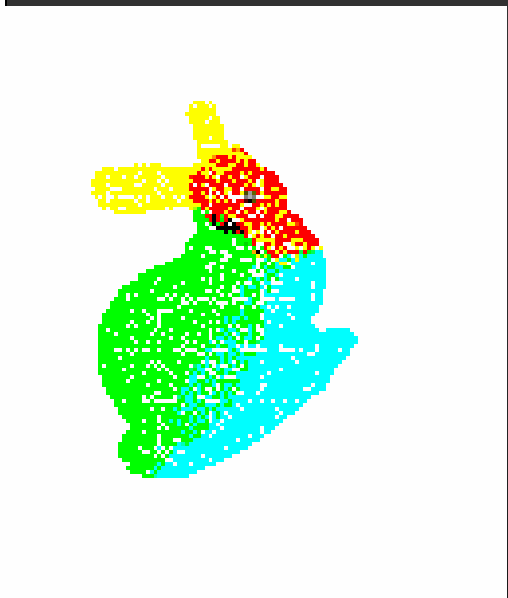
$$d_{\mathcal{GH}}(X,Y) \triangleq \inf_{Z,f,g} d_{\mathcal{H}}^{Z}(X,Y)$$

$f : X \to Z,\ g : Y \to Z$ *isometric embeddings*

Key question

- How to estimate the Gromov-Housdorff distance from noisy samples of the metric space





First step: Working with point clouds

Let X and Y be compact metric spaces, $\mathbf{X}_m$ an r-covering of X and $\mathbf{Y}_{m'}$ an r'-covering of Y. Then

$$|d_{\mathcal{GH}}(X,Y) - d_{\mathcal{GH}}(\mathbf{X}_m, \mathbf{Y}_{m'})| \leq r + r'$$

- **Consequence:** Working with point clouds "is possible"

How we compute the distance?

$$d_{\mathcal{I}}(\mathbf{X},\mathbf{Y}) \triangleq \min_{\pi\in\mathcal{P}_n} \max_{1\leq i,j\leq n} \frac{1}{2}|d_{\mathbf{X}}(x_i,x_j)-d_{\mathbf{Y}}(y_{\pi_i},y_{\pi_j})|$$

$$d_{\mathcal{GH}}(\mathbf{X},\mathbf{Y}) \leq d_{\mathcal{I}}(\mathbf{X},\mathbf{Y})$$

$$d_{\mathcal{GH}}(X,Y) \leq R_X + R_Y + d_{\mathcal{I}}(\mathbf{X},\mathbf{Y})$$

- **Consequence:** If we see a small pairwise distance, the objects are isometric

The need for a probabilistic framework

Let (X, d_X) and (Y, d_Y) be any pair of given compact metric spaces and let $\eta = d_{\mathcal{GH}}(X,Y)$. Also, let $N_{X,n}^{(r,s)} = \{x_1, \ldots, x_n\}$ be given. Then, given $\alpha > 0$ there exist points $\{y_1^\alpha, \ldots, y_n^\alpha\} \subset Y$ such that

1. $d_{\mathcal{I}}(N_{X,n}^{(r,s)}, \{y_1^\alpha, \ldots, y_n^\alpha\}) \leq (\eta + \alpha)$
2. $B_Y\Big(\{y_1^\alpha, \ldots, y_n^\alpha\}, r + 2(\eta + \alpha)\Big) = Y$
3. $d_Y(y_i^\alpha, y_j^\alpha) \geq s - 2(\eta + \alpha)$ for $i \neq j$.

The need for a probabilistic framework (cont.)

- The problem is well posed
- No reason for the *y*'s to be given:

$$\begin{aligned} d_{\mathcal{I}}(N_{X,n}^{(r,s)}, N_{Y,n}^{(\widehat{r},\widehat{s})}) &\le d_{\mathcal{I}}(N_{X,n}^{(r,s)}, N_{Y,n}^{(r,s)}) + d_{\mathcal{I}}(N_{Y,n}^{(\widehat{r},\widehat{s})}, N_{Y,n}^{(r,s)}) \\ &= 0 + \mathsf{small}(r, \widehat{r}) \end{aligned}$$

- We need probabilistic bounds!

The probabilistic framework

- ***Bottleneck distance*** **between two samples of the same space:**

$$d_{\mathcal{B}}^{Z}(\mathbf{Z},\mathbf{Z}') \stackrel{\triangle}{=} \min_{\pi\in\mathcal{P}_n} \max_{k} d_Z(z_k, z'_{\pi_k}) \geq d_{\mathcal{I}}(\mathbf{Z},\mathbf{Z}')$$

- **Using concepts from intrinsic Voronoi diagrams and coupon collector theorem we have:**

The probabilistic framework (cont.)

Let (Z, d_Z) be a smooth compact submanifold of $I\!R^d$. Given a covering $N_{Z,n}^{(r,s)}$ of Z and a number $p \in (0,1)$, there exists a positive integer $m = m_n(p)$ such that if $\mathbf{Z}_m = \{z_k\}_{k=1}^m$ is a sequence of $i.i.d.$ points sampled uniformly from Z, with probability p one can find a set of n different indices $\{i_1, \ldots, i_n\} \subset \{1, \ldots, m\}$ with

$$d_{\mathcal{B}}^Z(N_{Z,n}^{(r,s)}, \{z_{i_1}, \ldots, z_{i_n}\}) \leq r$$

The probabilistic framework (cont.)

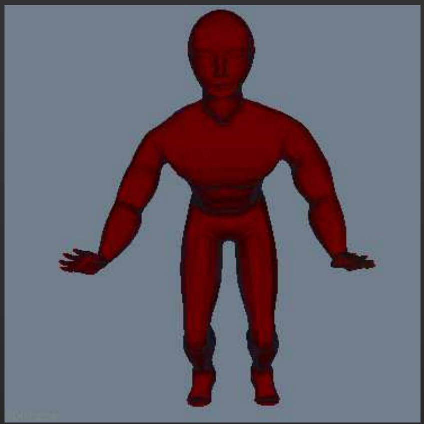
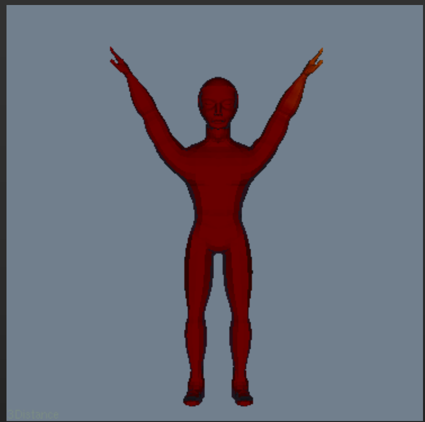
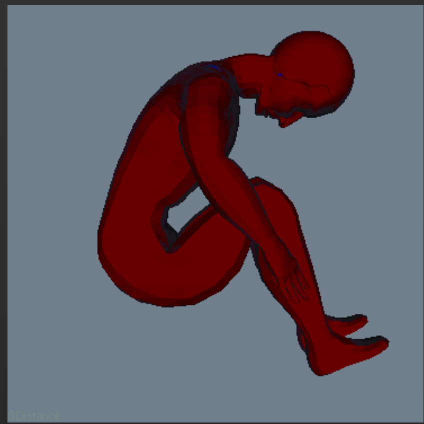
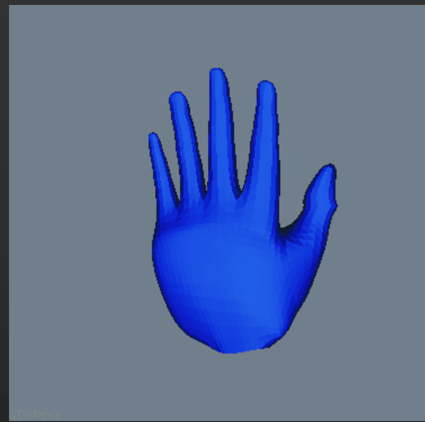
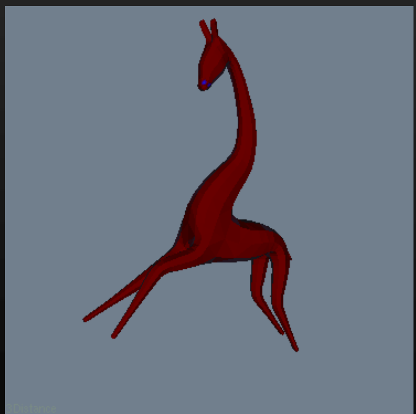
Let X and Y compact submanifolds of $\mathbb{R}^d$. Let $N^{(r,s)}_{X,n}$ be a covering of X with separation s such that for some positive constant c, $s - 2d_{\mathcal{GH}}(X,Y) > c$. Then, given any number $p \in (0,1)$, there exists a positive integer $m = m_n(p)$ such that if $\mathbf{Y}_m = \{y_k\}_{k=1}^m$ is a sequence of $i.i.d.$ points sampled uniformly from Y, we can find, with probability at least p, a set of n different indices $\{i_1, \ldots, i_n\} \subset \{1, \ldots, m\}$ such that

$$d_{\mathcal{I}}(N^{(r,s)}_{X,n}, \{y_{i_1}, \ldots, y_{i_n}\}) \leq 3\, d_{\mathcal{GH}}(X,Y) + r$$

Computational considerations

- Bounds on the number of sample points needed
- Covers of Y found using *farthest point sampling.*
- Geodesic distances for points on X and Y
- Select matching points of X and Y following our theory

Examples

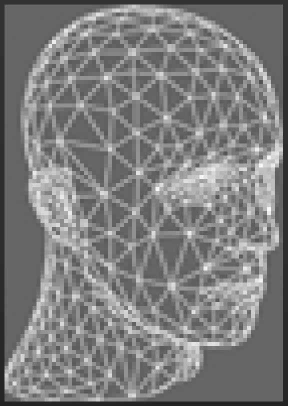
Meshless Geometric Subdivision

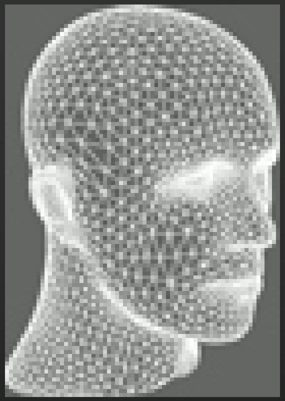
Joint with

Carsten Moenning
Facundo Memoli
Nira Dyn
N. Dodgson

What is and Motivation

- **Mesh based subdivision**
  - Refinement (add points and edges)
  - Averaging





- **Mesh not really geometric**

What is and Motivation (cont.)

- Point clouds are natural for 3D scanners
- Point clouds are the “true” geometry
- Point clouds are dimensionality independent
- All operations are geometric

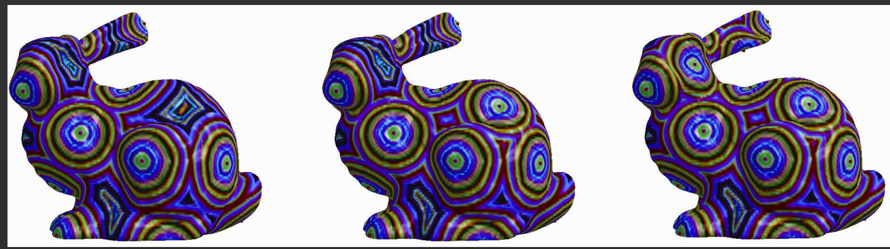
Main Steps

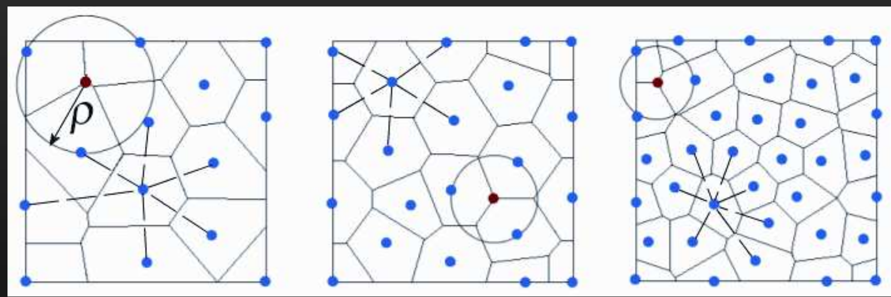
- Intrinsic point cloud simplification
- Intrinsic proximity information
- Geodesic centroid computation

- Intrinsic subdivision scheme

Intrinsic point cloud simplification

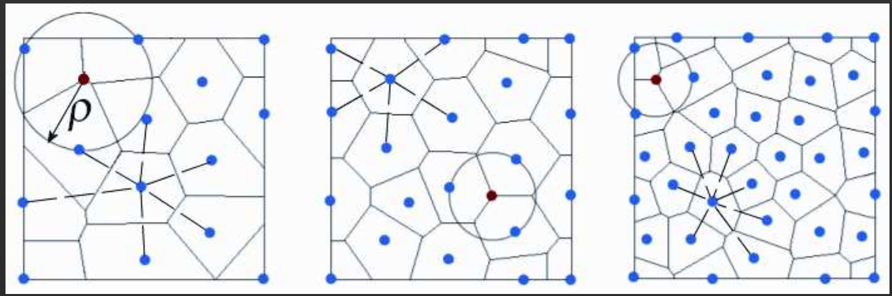
- Follows Meonning & Dodgson
- Based on progressive farthest point sampling
- Computed based on intrinsic Voronoi diagram (uses distance on point clouds)
- Guaranteed bounds on distance between samples

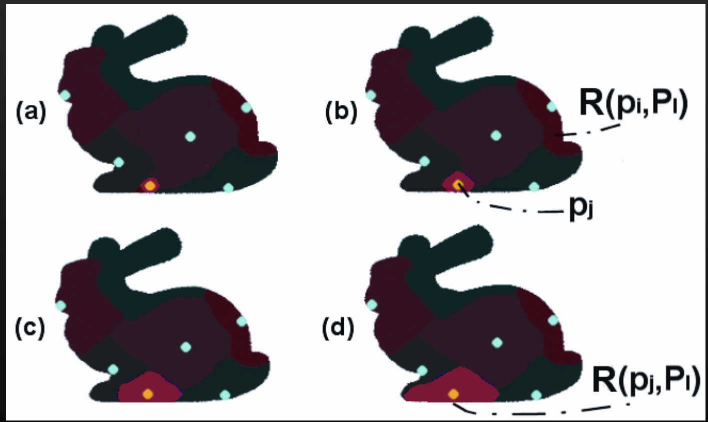




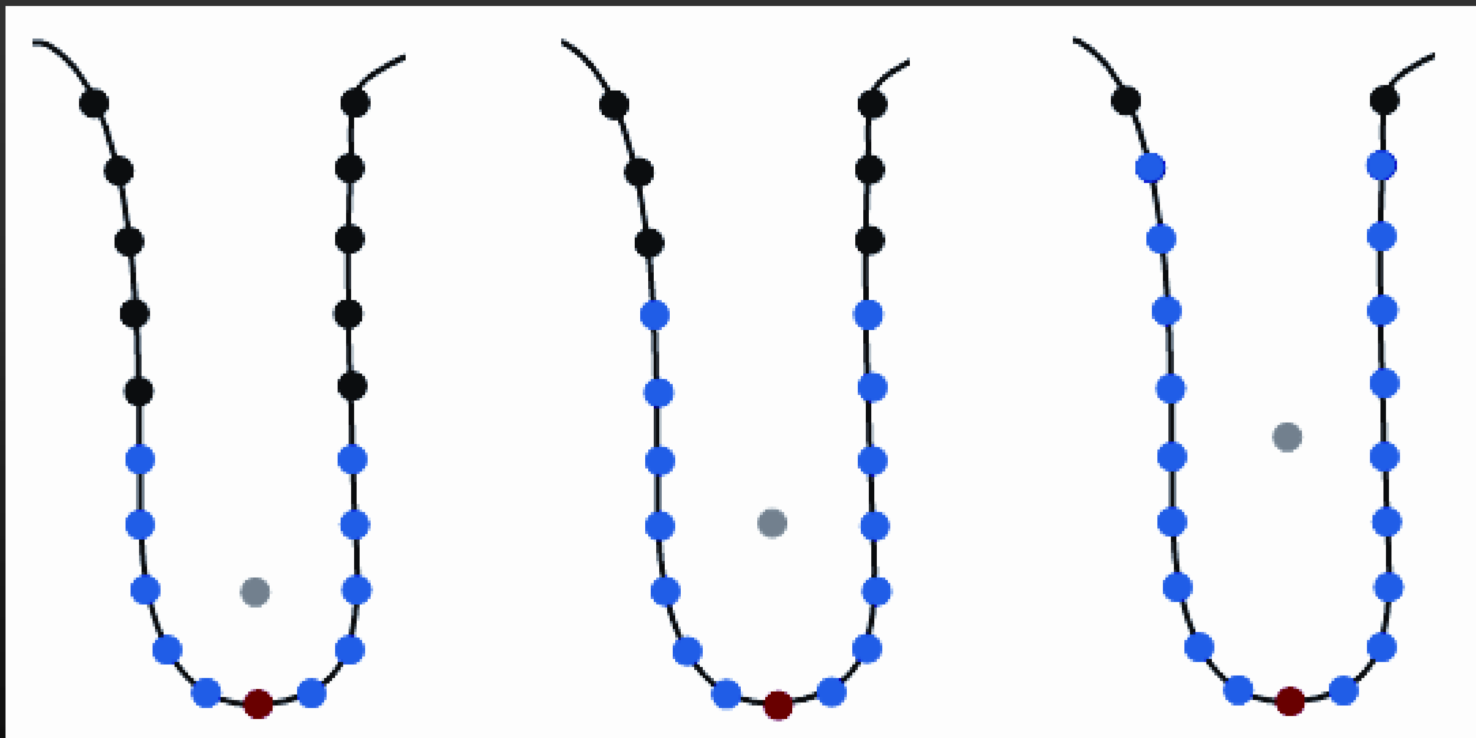
Intrinsic proximity information

- “Replaces” (non-geometric) connectivity in mesh techniques
- Given by neighbors from the intrinsic Voronoi
- Easily updated when the point cloud is refined (using geodesics on point clouds)





Geodesic centroid computation



Geodesic centroid computation (cont.)

$$\mathrm{centroid} := \min_g \frac{1}{2}\sum_{k=1}^{n} w_k d_M^2(g, p_k)$$

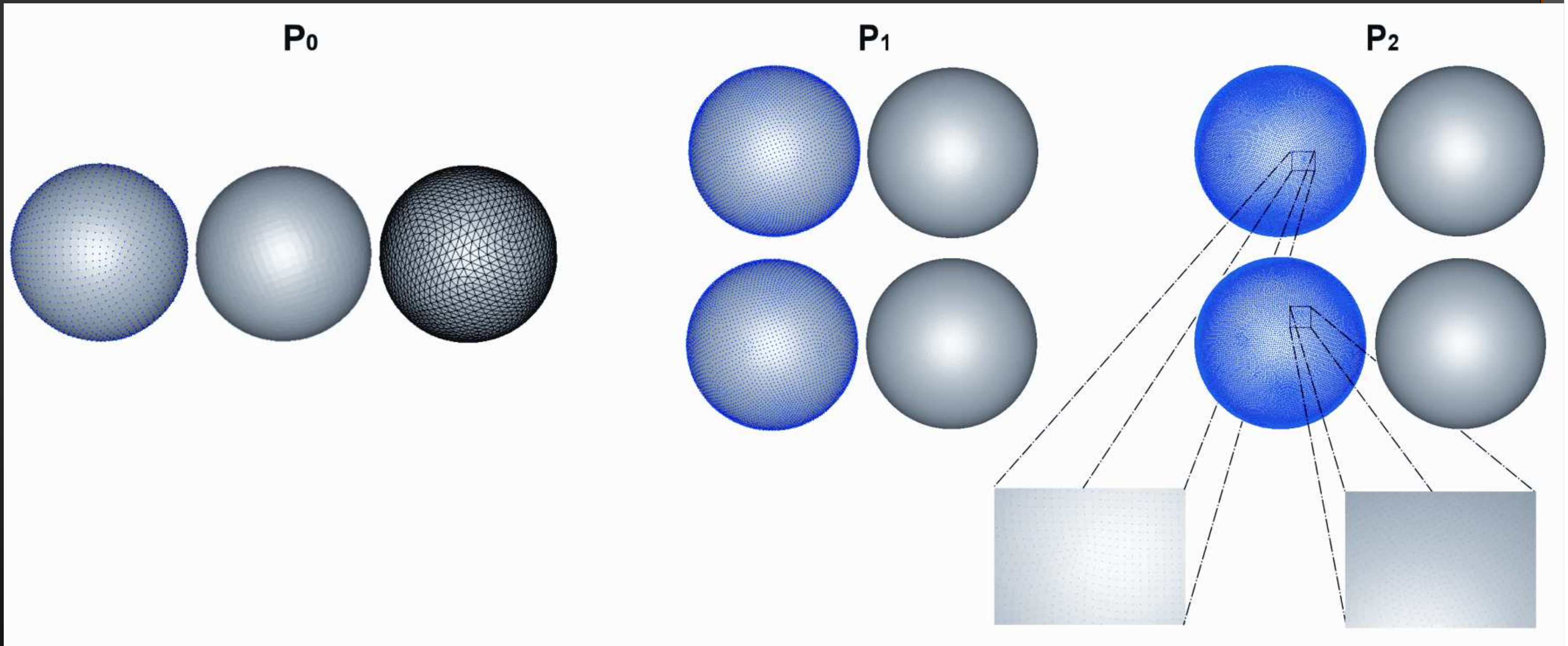
$$V(g) := \sum_{k=1}^{n} w_k \nabla_M \frac{1}{2} d_M^2(g, p_k) = 0$$

$$\left(g_0 := \Pi_M\left(\sum_{k=1}^{n} w_k p_k\right)\right) \rightarrow_{-V(g)} \mathrm{centroid}$$

Intrinsic subdivision scheme

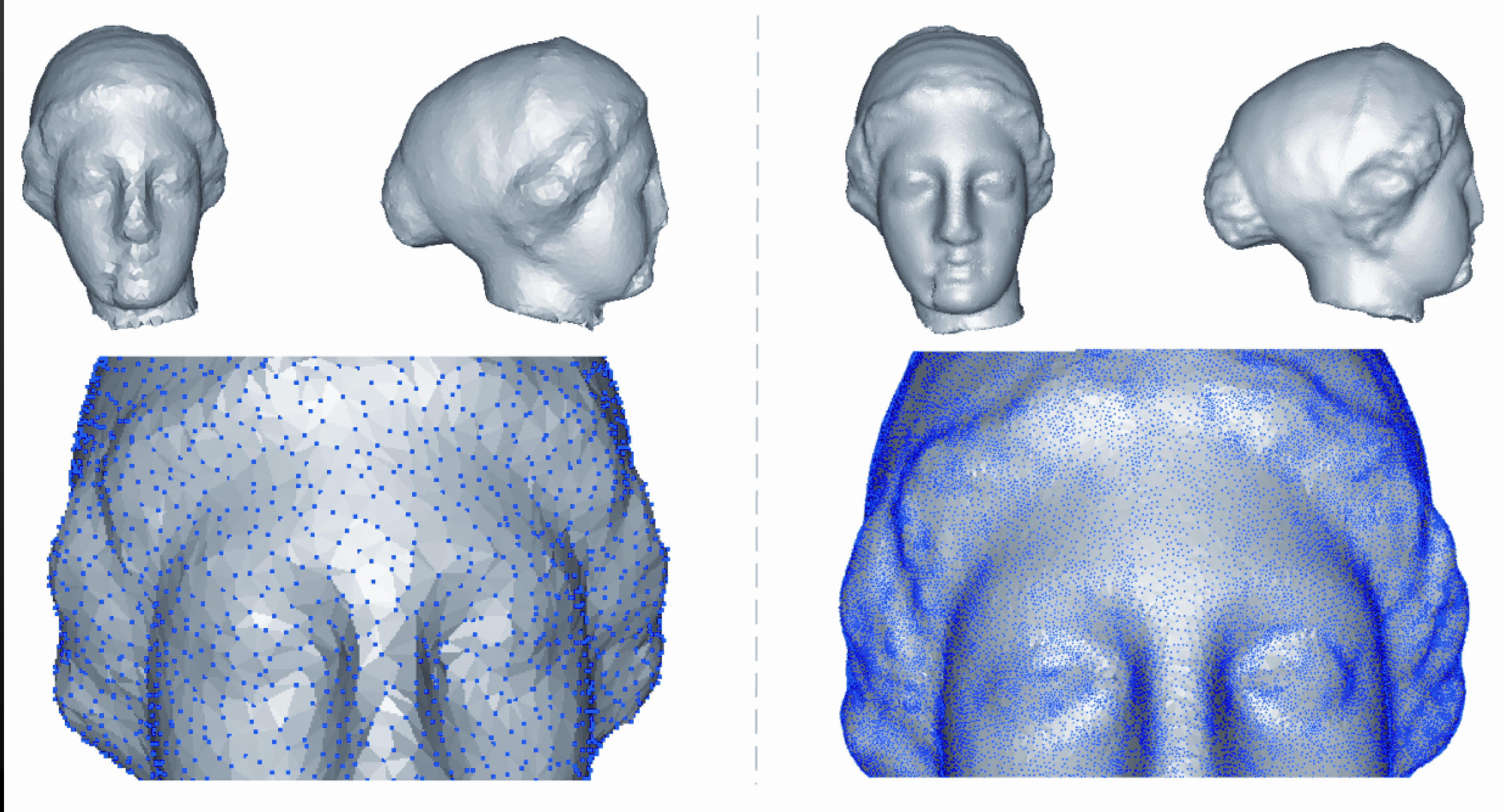
- **Geometric averaging rule:** Replace the point by the geodesic centroid of its intrinsic neighborhood
- **Refinement rule:** For each neighbor, insert the geodesic centroid of the joint neighborhood

Example



Quantitative study in the paper

Example



Conclusions

- Work with implicit surfaces and point clouds!!!

Thanks

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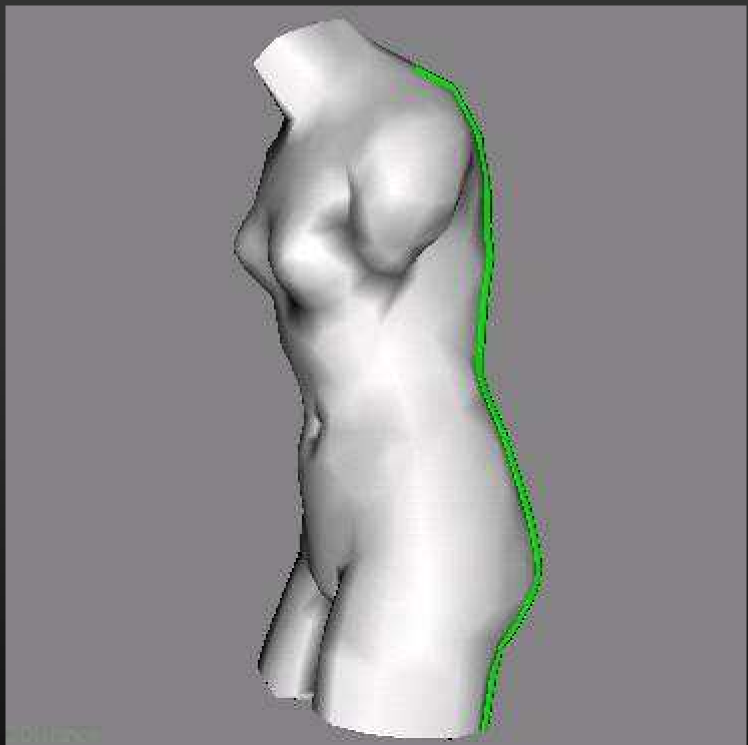
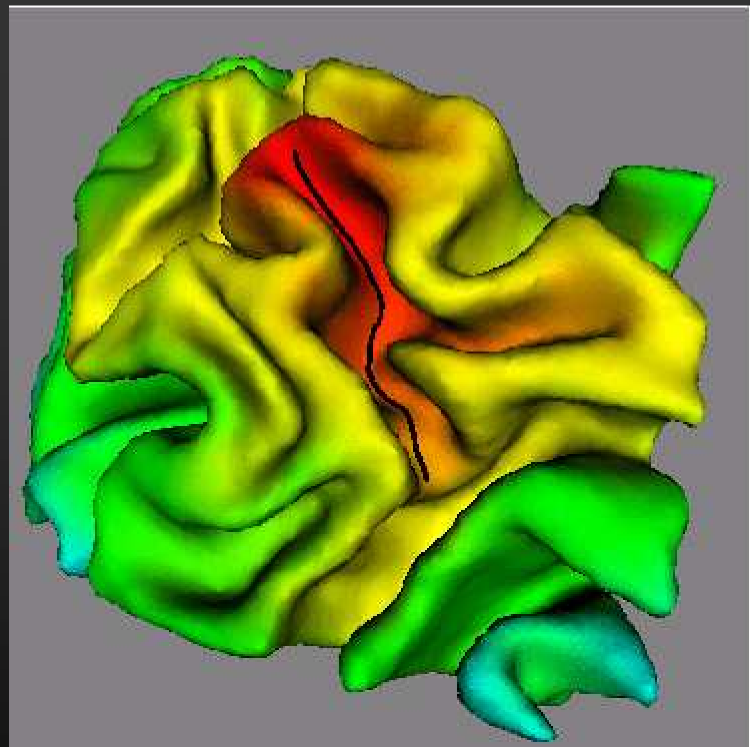
Sulcii extraction on meshes

(with A. Bartesaghi)





Follows Kimmel-Sethian