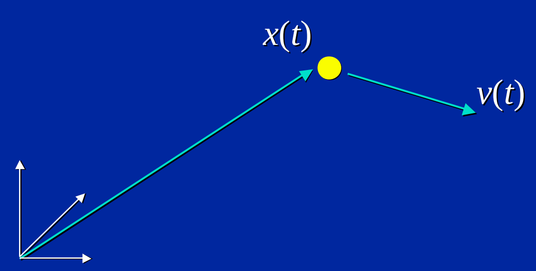
Rigid Body Simulation

Particle State

$$\mathbf{Y} = \begin{pmatrix} x(t) \\ v(t) \end{pmatrix}$$

$\mathbf{Y} =$	$x(t)$			$v(t)$		

Particle Motion

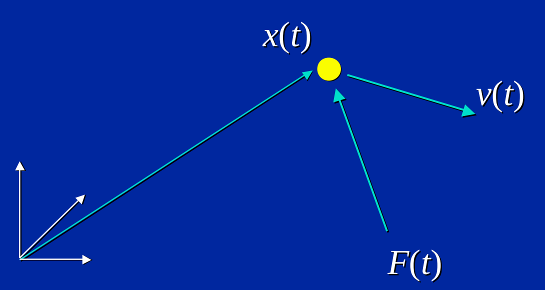


State Derivative

$$\frac{d}{dt}\mathbf{Y} = \frac{d}{dt}\begin{pmatrix} x(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} v(t) \\ F(t)/m \end{pmatrix}$$

$\frac{d}{dt}\mathbf{Y} =$	$v(t)$			$F(t)/m$		

Particle Dynamics

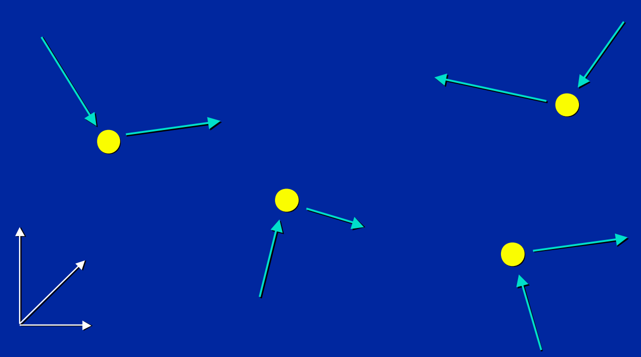


State Derivative

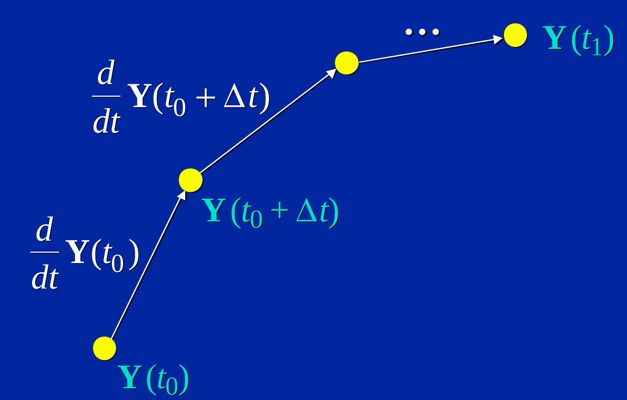
$$\frac{d}{dt}\mathbf{Y} = \frac{d}{dt}\begin{pmatrix} x_1(t) \\ v_1(t) \\ \\ x_n(t) \\ v_n(t) \end{pmatrix} = \begin{pmatrix} v_1(t) \\ F_1(t)/m_1 \\ \\ v_n(t) \\ F_n(t)/m_n \end{pmatrix}$$

$$\frac{d}{dt}\mathbf{Y} = \boxed{\quad}\boxed{\quad}\boxed{\cdots\ 6n \text{ elements}\ \cdots}\boxed{\quad}\boxed{\quad}$$

Multiple Particles



ODE solution



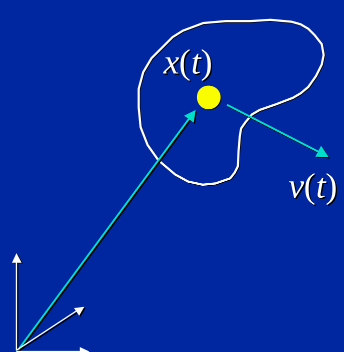
$\mathbf{Y}(t_0)$, `len`, t_0, t_1, `dydt` → ODE solver → $\mathbf{Y}(t_1)$

```
void dydt(double t, double y[],
                    double ydot[])
```

dydt

$$\mathbf{Y}(t) = \begin{pmatrix} x_1(t) \\ v_1(t) \\ \vdots \\ x_n(t) \\ v_n(t) \end{pmatrix} \qquad \frac{d}{dt}\mathbf{Y}(t) = \begin{pmatrix} v_1(t) \\ F_1(t)/m_1 \\ \vdots \\ v_n(t) \\ F_n(t)/m_n \end{pmatrix}$$

Rigid Body State

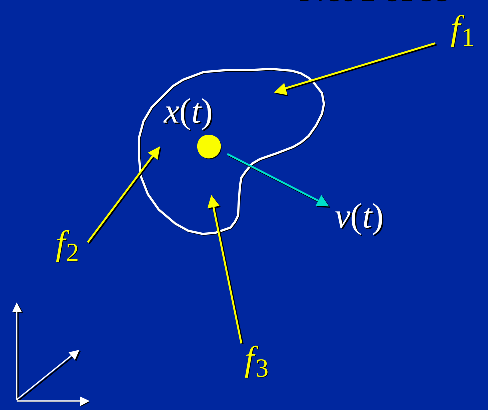


$$\mathbf{Y} = \begin{pmatrix} x(t) \\ ? \\ v(t) \\ ? \end{pmatrix}$$

Rigid Body Equation of Motion

$$\frac{d}{dt}\mathbf{Y} = \frac{d}{dt}\begin{pmatrix} x(t) \\ ? \\ Mv(t) \\ ? \end{pmatrix} = \begin{pmatrix} v(t) \\ ? \\ F(t) \\ ? \end{pmatrix}$$

Net Force



$$F(t) = \sum f_i$$

Orientation

We represent orientation as a rotation matrix† $R(t)$. Points are transformed from body-space to world-space as:

$$p(t) = R(t)p_0 + x(t)$$

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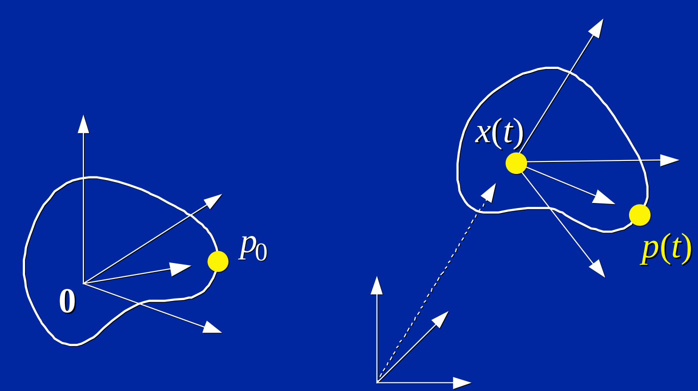
†He's lying. Actually, we use quaternions.



body space

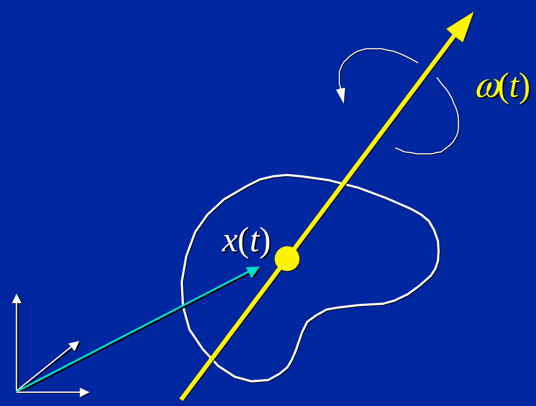
world space

Angular Velocity

We represent angular velocity as a vector $\omega(t)$, which encodes both the axis of the spin and the speed of the spin.

How are $R(t)$ and $\omega(t)$ related?

Angular Velocity Definition



Angular Velocity

$\dot{R}(t)$ and $\omega(t)$ are related by

$$\frac{d}{dt}R(t) = \begin{pmatrix} 0 & -\omega_z(t) & \omega_y(t) \\ \omega_z(t) & 0 & -\omega_x(t) \\ -\omega_y(t) & \omega_x(t) & 0 \end{pmatrix} R(t)$$

($\omega(t)^*$ is a shorthand for the above matrix)

Rigid Body Equation of Motion

$$\frac{d}{dt}\mathbf{Y} = \frac{d}{dt}\begin{pmatrix} x(t) \\ R(t) \\ Mv(t) \\ < \omega(t) > \end{pmatrix} = \begin{pmatrix} v(t) \\ \omega(t)^* R(t) \\ F(t) \\ ? \end{pmatrix}$$

Need to relate $\dot{\omega}(t)$ and mass distribution to $F(t)$.

Inertia Tensor

$$I(t) = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$

diagonal terms

$$I_{xx} = M \int_V (y^2 + z^2)\, dV$$

off-diagonal terms

$$I_{xy} = -M \int_V xy\, dV$$

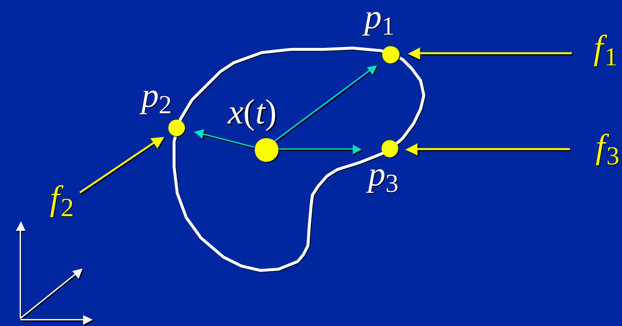
Rigid Body Equation of Motion

$$\frac{d}{dt}\mathbf{Y} = \frac{d}{dt}\begin{pmatrix} x(t) \\ R(t) \\ \boxed{Mv(t)} \\ \boxed{I(t)\omega(t)} \end{pmatrix} = \begin{pmatrix} v(t) \\ \omega(t)^{*} R(t) \\ F(t) \\ \tau(t) \end{pmatrix}$$

$P(t)$ – linear momentum

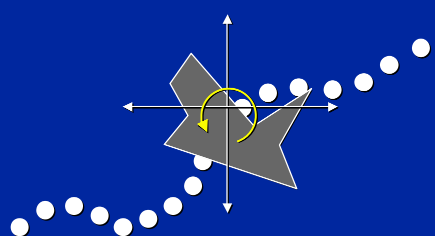
$L(t)$ – angular momentum

Net Torque



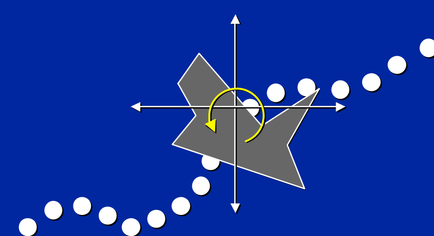
$$\tau(t) = \sum (p_i - x(t)) \times f_i$$

Inertia Tensors Vary in World Space...



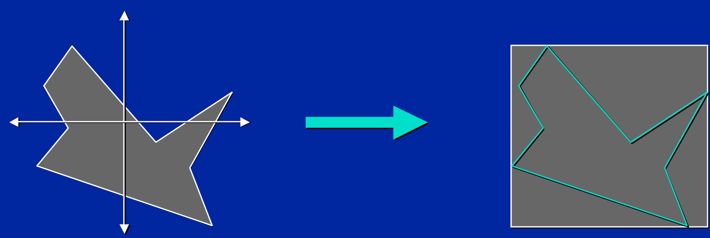
$$I_{xx} = M\int_V (y^2 + z^2)\, dV \qquad I_{xy} = -M\int_V xy\, dV$$

... but are Constant in Body Space



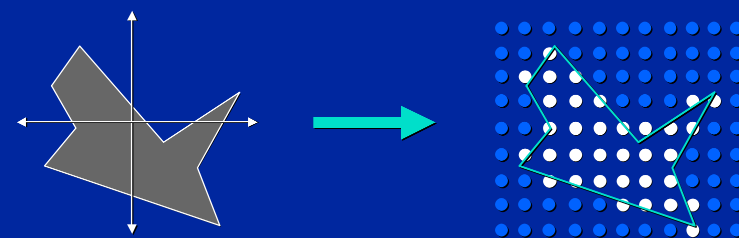
$$I(t) = R(t) I_{\text{body}} R(t)^T$$

Approximating I_{body}—Bounding Boxes



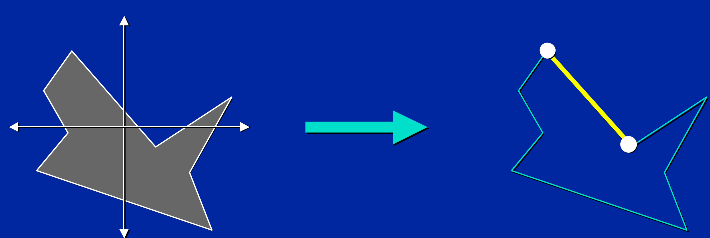
Pros: Simple.
Cons: Bounding box may not be a good fit. Inaccurate.

Approximating I_{body}—Point Sampling



Pros: Simple, fairly accurate, no B-rep needed.
Cons: Expensive, requires volume test.

Computing I_{body}— Green's Theorem (Twice!)



Pros: Simple, exact, no volumes needed.
Cons: Requires B-rep.
Code: http://www.acm.org/jgt/papers/Mirtich96

Computing I_{body} at the center of mass

The inertia tensor should be relative to the body's center of mass, e.g:

$$I_{xx} = I'_{xx} - m\left(r_y{}^2 + r_z{}^2\right)$$

$$I_{xy} = I'_{xy} - m r_x r_y$$

Rigid Body Equation of Motion

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$P(t)$ – linear momentum

$L(t)$ – angular momentum