Dynamics of Transformation Hierarchies

Kinetic Energy

\[ T_j = \frac{1}{2} \int \mathbf{p}_j^T \mathbf{p}_j \tau_i \, dx \, dy \, dz \]
\[ = \frac{1}{2} \int \mathbf{x}_j^T \mathbf{W}_j^T \mathbf{W}_j \mathbf{x}_i \tau_i \, dx \, dy \, dz \]
\[ = \frac{1}{2} \int tr\left( \mathbf{W}_j \mathbf{x}_i^T \mathbf{x}_i^T \tau_i \, dx \, dy \, dz \right) \mathbf{W}_j^T \tau_i \, dx \, dy \, dz \]
\[ = \frac{1}{2} tr\left( \mathbf{W}_j \mathbf{M}_i \mathbf{W}_j^T \right) \]

where \( \mathbf{M}_i \) is the primitive mass tensor.

Kinetic Energy Lagrangian Contribution

\[ \frac{d}{dt} \frac{\partial T_i}{\partial \dot{q}_j} - \frac{\partial T_i}{\partial q_j} = \frac{1}{2} tr\left( \frac{\partial \mathbf{W}_i}{\partial q_j} \mathbf{M}_i \mathbf{W}_i^T + \dot{\mathbf{W}}_i \frac{\partial \mathbf{W}_i}{\partial q_j} \right) \]
\[ = tr\left( \frac{\partial \mathbf{W}_i}{\partial q_j} \mathbf{M}_i \dot{\mathbf{W}}_i^T \right) \]

Gereralized Force Equation

\[ C_j = \sum_i \left[ tr\left( \frac{\partial \mathbf{W}_i}{\partial q_j} \mathbf{M}_i \dot{\mathbf{W}}_i^T \right) + m_i g \frac{\partial \mathbf{W}_i}{\partial q_j} \mathbf{c}_i \right] \]
\[ + \sum_k \left[ \frac{\partial (\mathbf{F}_k \mathbf{p}_k)}{\partial q_j} \right] \]
\[ + \sum_l \left[ \lambda_l \frac{\partial \mathbf{C}_l}{\partial q_j} \right] \]

where \( i \) ranges over all primitives, \( k \) over all point forces, and \( l \) over all mechanical constraints.
Recursive Formulation

$S(i)$ is the set of all node indices in the subtree rooted at node $i$.

$$\sum_i tr \left( \frac{\partial W_i}{\partial q_j} M_j W_i^T \right) + m_i g \frac{\partial W_i}{\partial q_j} c_i$$

$$= \sum_{i \in S(j)} \left( \frac{\partial W_i}{\partial q_j} M_i W_i^T \right) + m_i g \frac{\partial W_i}{\partial q_j} c_i$$

$$= tr \left( \frac{\partial W_j}{\partial q_j} \sum_{i \in S(j)} W_i^j M_j W_i^T \right)$$

$$+ g \frac{\partial W_j}{\partial q_j} \sum_{i \in S(j)} m_i W_i^j c_i.$$

Recursive Formulation

We introduce two new recursively defined variables

$$\dot{c}_i = m_i c_i + \sum_{j \in S(i)} R_{ji} \ddot{e}_j$$

$$\ddot{M}_i = M_i \ddot{W}_i^T + \sum_{j \in S(i)} R_{ji} \ddot{M}_j.$$

Finally, recursive definition of the Newtonian constraint

$$C_j = tr \left( \frac{\partial W_j}{\partial q_j} \dot{M}_j \right) + \frac{\partial W_j}{\partial q_j} \dot{e}_j$$

$$+ \sum_k \left[ \frac{\partial F_k}{\partial q_j} p_k \right] + \sum_l \left[ q_{jl} \frac{\partial C_{mj}}{\partial q_j} \right].$$