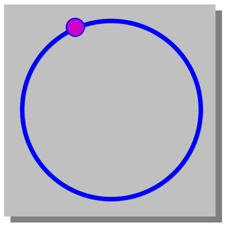
Differential Constraints

Beyond Points and Springs

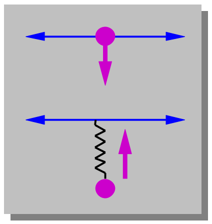
- **You can make just about anything out of point masses and springs, *in principle***
- **In practice, you can make anything you want as long as it's jello**
- **Constraints will buy us:**
  - **Rigid links instead of goopy springs**
  - **Ways to make interesting contraptions**

A bead on a wire



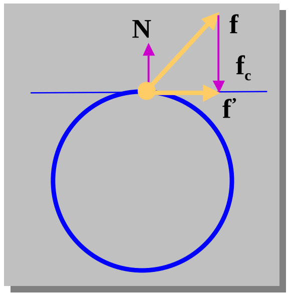
- **Desired Behavior:**
  - **The bead can slide freely *along* the circle**
  - **It can never come off, however hard we pull**
- **Question:**
  - **How does the bead move under applied forces?**

Penalty Constraints



- **Why not use a spring to hold the bead on the wire?**
- **Problem:**
  - **Weak springs ⇒ goopy constraints**
  - **Strong springs ⇒ neptune express!**
- **A classic *stiff system***

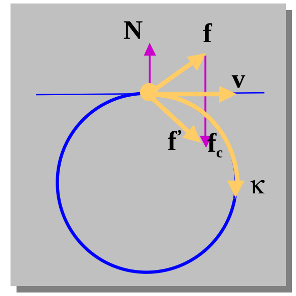
The basic trick (f = mv version)



- 1st order world.
- *Legal velocity:* tangent to circle (N·v = 0)
- *Project* applied force f onto tangent: f' = f + f_c
- Added normal-direction force f_c: *constraint force*
- No tug-of-war, no stiffness

$$\mathbf{f}_c = -\frac{\mathbf{f}\cdot\mathbf{N}}{\mathbf{N}\cdot\mathbf{N}}\mathbf{N} \qquad \mathbf{f}' = \mathbf{f} + \mathbf{f}_c$$

f = ma

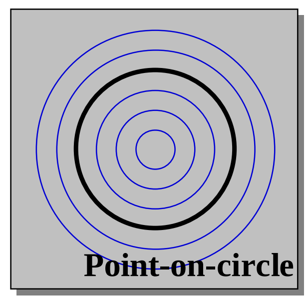


- Same idea, but…
- *Curvature* (κ) has to match.
- κ depends on *both* a and v:
  - the faster you're going, the faster you have to turn
- Calculate f_c to yield a legal *combination* of a and v
- Not as simple!

Now for the Algebra …

- Fortunately, there's a general recipe for calculating the constraint force
- First, a single constrained particle
- Then, generalize to constrained particle systems

Representing Constraints



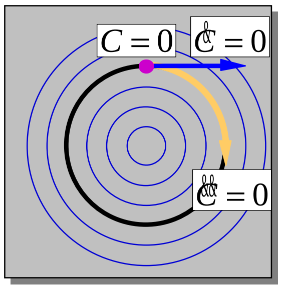
I. Implicit:

$$C(\mathbf{x}) = |\mathbf{x}| - r = 0$$

~~*II. Parametric:*~~

~~$\mathbf{x} = r\,[\cos\theta, \sin\theta]$~~

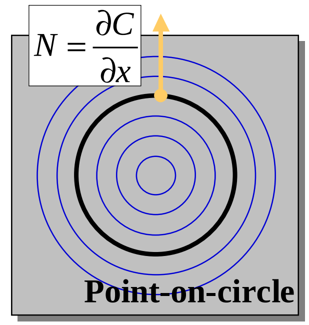
Maintaining Constraints Differentially



- **Start with legal position and velocity.**
- **Use constraint forces to ensure legal curvature.**

$C = 0$	legal position
$\dot{C} = 0$	legal velocity
$\ddot{C} = 0$	legal curvature

Constraint Gradient



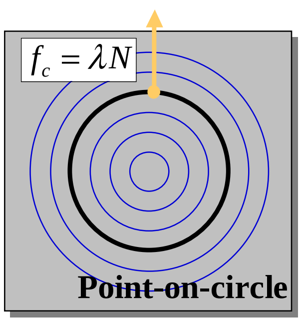
Implicit:

$C(\mathbf{x}) = |\mathbf{x}| - r = 0$

Differentiating C gives a normal vector.

This is the direction our constraint force will point in.

Constraint Forces



Constraint force: gradient vector times a scalar λ

Just one unknown to solve for

Assumption: constraint is passive—no energy gain or loss

Constraint Force Derivation

$C(x(t))$

$\dot{C} = N \cdot \dot{x}$

$\ddot{C} = \frac{\partial}{\partial t}(N \cdot x)$

$= \dot{N} \cdot \dot{x} + N \cdot \ddot{x}$

$\ddot{x} = \frac{f + f_c}{m}$

$f_c = \lambda N$

Set $\ddot{C}$ = 0, solve for λ:

$\lambda = -m \frac{\dot{N} \cdot \dot{x}}{N \cdot N} - \frac{N \cdot f}{N \cdot N}$

Constraint force is λN.

Notation: $N = \frac{\partial C}{\partial x}, \dot{N} = \frac{\partial^2 C}{\partial x \partial t}$

Example: Point-on-circle

$$C = |\mathbf{x}| - r$$

$$\mathbf{N} = \frac{\partial C}{\partial \mathbf{x}} = \frac{\mathbf{x}}{|\mathbf{x}|}$$

$$\dot{\mathbf{N}} = \frac{\partial^2 C}{\partial \mathbf{x} \partial t} = \frac{1}{|\mathbf{x}|}\left[\dot{\mathbf{x}} - \frac{\mathbf{x}\cdot\dot{\mathbf{x}}}{\mathbf{x}\cdot\mathbf{x}}\mathbf{x}\right]$$

Write down the constraint equation.

Take the derivatives.

Substitute into generic template, simplify.

$$\lambda = -m\frac{\dot{\mathbf{N}}\cdot\dot{\mathbf{x}}}{\mathbf{N}\cdot\mathbf{N}} - \frac{\mathbf{N}\cdot\mathbf{f}}{\mathbf{N}\cdot\mathbf{N}} = \left[m\frac{(\mathbf{x}\cdot\dot{\mathbf{x}})^2}{\mathbf{x}\cdot\mathbf{x}} - m(\dot{\mathbf{x}}\cdot\dot{\mathbf{x}}) - \mathbf{x}\cdot\mathbf{f}\right]\frac{1}{|\mathbf{x}|}$$

Drift and Feedback

- **In principle, clamping $\ddot{C}$ at zero is enough**
- **Two problems:**
  - **Constraints might not be met initially**
  - **Numerical errors can accumulate**
- **A feedback term handles both problems:**

$$\ddot{C} = -\alpha C - \beta\dot{C}, \text{ instead of}$$
$$\ddot{C} = 0$$

α and β are magic constants.

Tinkertoys

- **Now we know how to simulate a bead on a wire.**
- **Next: a constrained particle *system*.**
  - **E.g. constrain particle/particle distance to make rigid links.**
- **Same idea, but…**

Constrained particle systems

- **Particle system: a point in state space.**
- **Multiple constraints:**
  - **each is a function $C_i(x_1, x_2, \ldots)$**
  - ***Legal state:* $C_i = 0, \forall i$.**
  - ***Simultaneous* projection.**
  - **Constraint force: *linear combination* of constraint gradients.**
- **Matrix equation.**

Compact Particle System Notation

$$\ddot{q} = WQ$$

q: ***3n*-long *state vector.***

Q: ***3n*-long *force vector.***

M: ***3n* x *3n* diagonal *mass matrix.***

W: M-inverse (element- wise reciprocal)

$$q = [x_1, x_2, \cdots, x_n]$$

$$Q = [f_1, f_2, \cdots, f_n]$$

$$M = \begin{bmatrix} m_1 & & & & & & \\ & m_1 & & & & & \\ & & m_1 & & & & \\ & & & \ddots & & & \\ & & & & m_n & & \\ & & & & & m_n & \\ & & & & & & m_n \end{bmatrix}$$

$$W = M^{-1}$$

Particle System Constraint Equations

Matrix equation for λ

$$[JWJ^T]\lambda = -\dot{J}\dot{q} - [JW]Q$$

Constrained Acceleration

$$\ddot{q} = W[Q + J^T\lambda]$$

Derivation: just like bead-on-wire.

More Notation

$$C = [C_1, C_2, \cdots, C_m]$$

$$\lambda = [\lambda_1, \lambda_2, \cdots, \lambda_m]$$

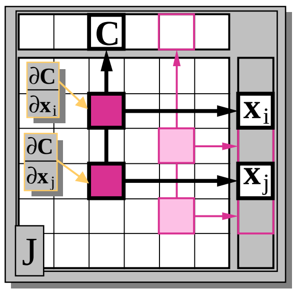
$$J = \frac{\partial C}{\partial q}$$

$$\dot{J} = \frac{\partial^2 C}{\partial q \partial t}$$

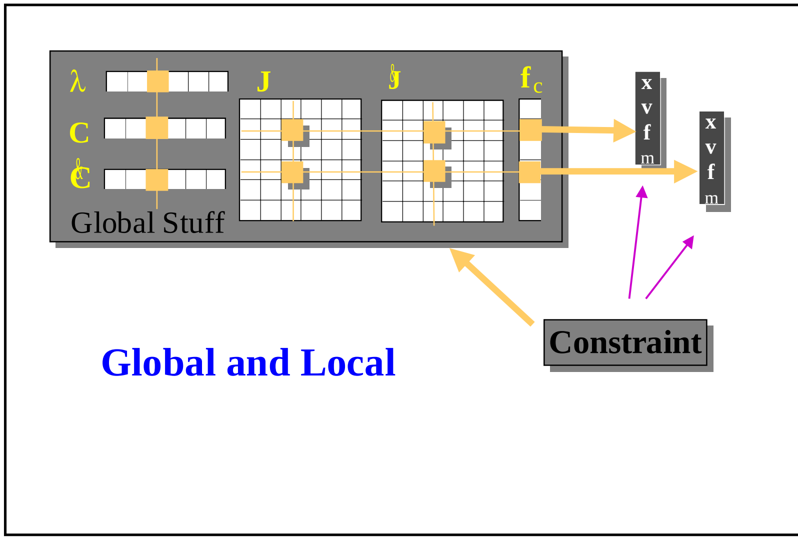
How do you implement all this?

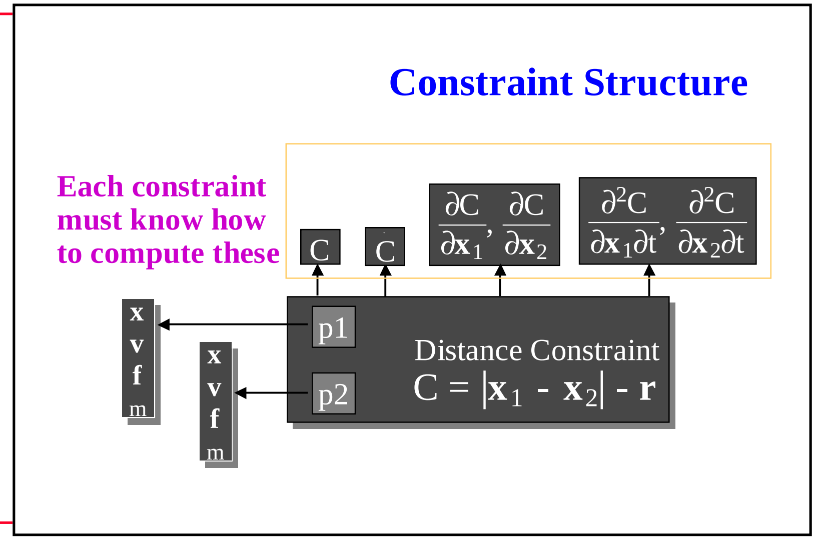
- **We have a global matrix equation.**
- **We want to build models on the fly, just like masses and springs.**
- **Approach:**
  - **Each constraint adds its own piece to the equation.**

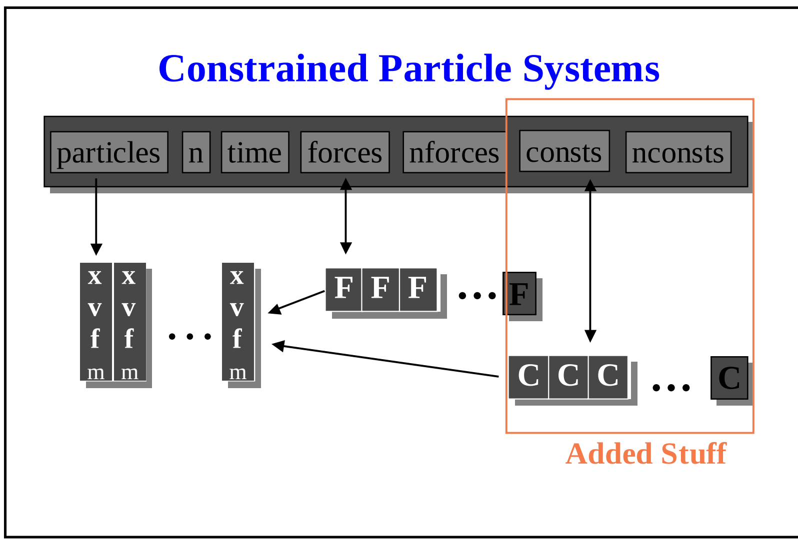
Matrix Block Structure

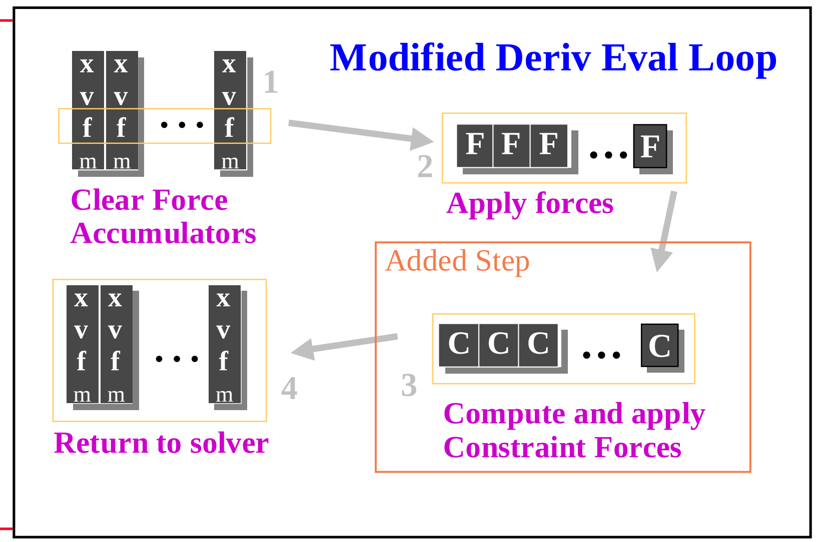


- **Each constraint contributes one or more *blocks* to the matrix.**
- **Sparsity: many empty blocks.**
- **Modularity: let each constraint compute its own blocks.**
- **Constraint and particle indices determine block locations.**


λ
C
Ċ
J
J̇
f_c
x v f m
Global Stuff
Constraint
Global and Local


Constraint Structure
Each constraint must know how to compute these
C
Ċ
$\frac{\partial C}{\partial \mathbf{x}_1}, \frac{\partial C}{\partial \mathbf{x}_2}$
$\frac{\partial^2 C}{\partial \mathbf{x}_1 \partial t}, \frac{\partial^2 C}{\partial \mathbf{x}_2 \partial t}$
x v f m
p1
p2
Distance Constraint
$C = |\mathbf{x}_1 - \mathbf{x}_2| - \mathbf{r}$


Constrained Particle Systems
particles
n
time
forces
nforces
consts
nconsts
x v f m
F F F … F
C C C … C
Added Stuff


Modified Deriv Eval Loop
x v f m
1
Clear Force Accumulators
2
F F F … F
Apply forces
Added Step
3
C C C … C
Compute and apply Constraint Forces
4
Return to solver

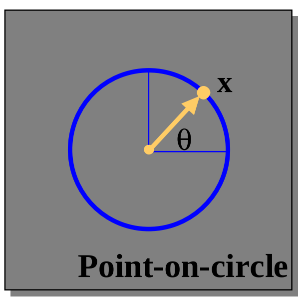
Constraint Force Eval

- After computing ordinary forces:
  - Loop over constraints, assemble global matrices and vectors.
  - Call matrix solver to get λ, multiply by J^T to get constraint force.
  - Add constraint force to particle force accumulators.

Impress your Friends

- The requirement that constraints not add or remove energy is called the *Principle of Virtual Work.*
- The λ's are called *Lagrange Multipliers.*
- The derivative matrix, J, is called the *Jacobian Matrix.*

A whole other way to do it.



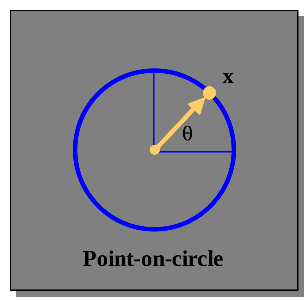
~~*I. Implicit:*~~

~~$C(\mathbf{x}) = |\mathbf{x}| - r = 0$~~

II. Parametric:

$\mathbf{x} = r\left[\cos\theta, \sin\theta\right]$

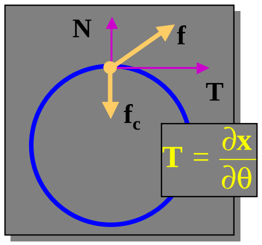
Parametric Constraints



Parametric:
$\mathbf{x} = r\left[\cos\theta, \sin\theta\right]$

- Constraint is always met exactly.
- One DOF: θ.
- Solve for $\ddot{\theta}$.

Parametric bead-on-wire (f = mv)



x is not an independent variable.

First step—get rid of it:

$$\ddot{x} = \frac{f + f_c}{m}$$ **f = mv *(constrained)***

$$\ddot{x} = T\ddot{\theta}$$ **chain rule**

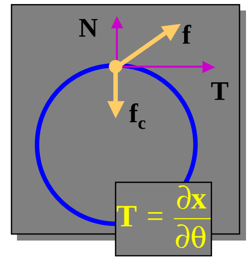
$$T\ddot{\theta} = \frac{f + f_c}{m}$$ **combine**

For our next trick…

As before, assume f_c points in the normal direction, so

$$T \cdot f_c = 0$$

We can nuke f_c by dotting T into both sides:



$$T\ddot{\theta} = \frac{f + f_c}{m}$$ **from last slide**

$$T \cdot T\ddot{\theta} = \frac{T \cdot f + T \cdot f_c}{m}$$ **blam!**

$$\ddot{\theta} = \frac{1}{m}\frac{T \cdot f}{T \cdot T}$$ **rearrange.**

General case

Lagrange dynamics:

$$\mathbf{J}^T\mathbf{M}\mathbf{J}\ddot{\mathbf{u}} + \mathbf{J}^T\mathbf{M}\dot{\mathbf{J}}\dot{\mathbf{u}} - \mathbf{J}^T\mathbf{Q} = 0$$

where

$$\mathbf{J} = \frac{\partial \mathbf{q}}{\partial \mathbf{u}}$$

Not to be confused with:

$$\left[\mathbf{J}\mathbf{W}\mathbf{J}^T\right]\lambda = -\dot{\mathbf{J}}\dot{\mathbf{q}} - [\mathbf{J}\mathbf{W}]\mathbf{Q}$$

where

$$\mathbf{J} = \frac{\partial \mathbf{C}}{\partial \mathbf{q}}$$

Parametric Constraints: Summary

- **Generalizations: f = ma, particle systems**
  - **Like implicit case (see notes.)**
- **Big advantages:**
  - **Fewer DOF's.**
  - **Constraints are always met.**
- **Big *dis*advantages:**
  - **Hard to formulate constraints.**
  - **No easy way to *combine* constraints.**
- **Offical name: *Lagrangian dynamics*.**

Hybrid systems

$$\left[\mathbf{J}\mathbf{W}\mathbf{J}^T\right]\boldsymbol{\lambda} = -\dot{\mathbf{J}}\dot{\mathbf{u}} - [\mathbf{J}\mathbf{W}]\mathbf{Q}$$

where

$$\mathbf{W} = \mathbf{M}^{-1} = \left[\iiint_i m_i \mathbf{q}_i{}^T \mathbf{q}_i\right]^{-1}$$

$$\mathbf{C}(\mathbf{q}(\mathbf{u}))$$

$$\mathbf{J} = \frac{\partial \mathbf{C}}{\partial \mathbf{q}} \frac{\partial \mathbf{q}}{\partial \mathbf{u}}$$

Project 1:

- **A bead on a wire (implicit)**
- **A double pendulum**
- **A *triple* pendulum**
- **Simple interactive tinkertoys**