

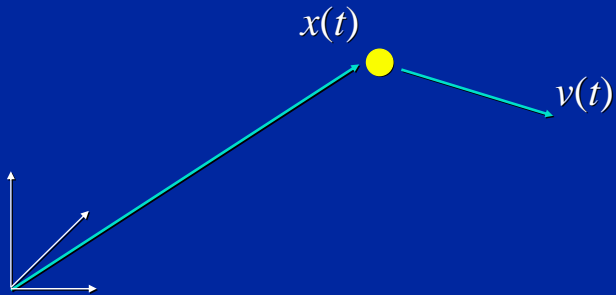
Rigid Body Simulation

Particle State

$$\mathbf{Y} = \begin{pmatrix} x(t) \\ v(t) \end{pmatrix}$$

$$\mathbf{Y} = \begin{array}{|c|c|c|c|c|c|} \hline & \overbrace{\hspace{2cm}}^{x(t)} & & \overbrace{\hspace{2cm}}^{v(t)} & & \\ \hline \square & \square & \square & \square & \square & \square \\ \hline \end{array}$$

Particle Motion

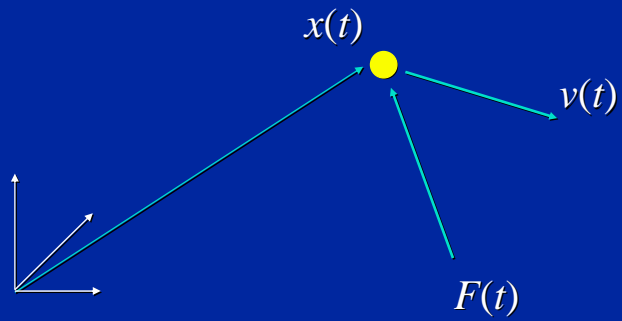


State Derivative

$$\frac{d}{dt} \mathbf{Y} = \frac{d}{dt} \begin{pmatrix} x(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} v(t) \\ F(t)/m \end{pmatrix}$$

$$\frac{d}{dt} \mathbf{Y} = \begin{array}{|c|c|c|c|c|c|} \hline & \overbrace{\hspace{2cm}}^{v(t)} & & \overbrace{\hspace{2cm}}^{F(t)/m} & & \\ \hline \square & \square & \square & \square & \square & \square \\ \hline \end{array}$$

Particle Dynamics

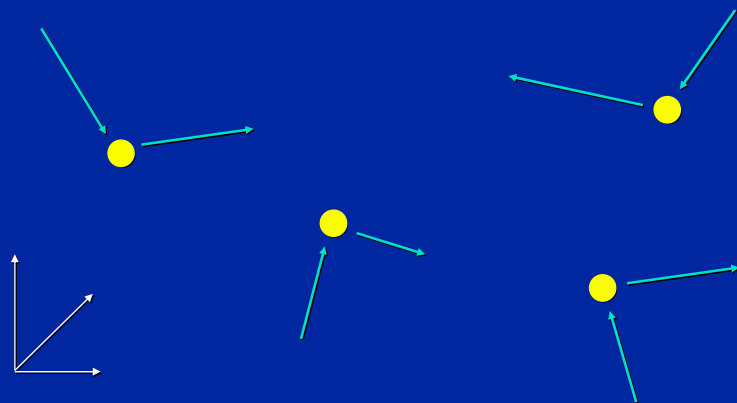


State Derivative

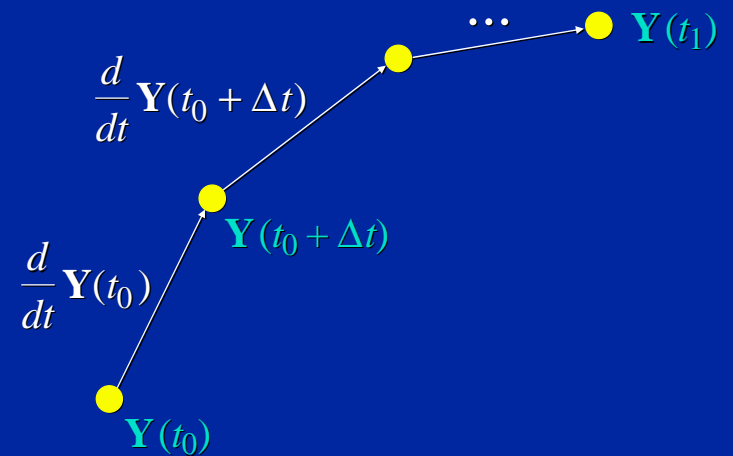
$$\frac{d}{dt} \mathbf{Y} = \frac{d}{dt} \begin{pmatrix} x_1(t) \\ v_1(t) \\ \vdots \\ x_n(t) \\ v_n(t) \end{pmatrix} = \begin{pmatrix} v_1(t) \\ F_1(t)/m_1 \\ \vdots \\ v_n(t) \\ F_n(t)/m_n \end{pmatrix}$$

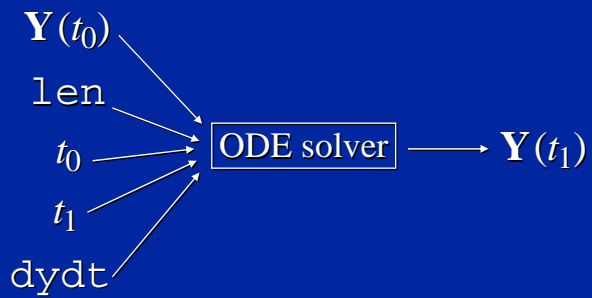
$$\frac{d}{dt} \mathbf{Y} = \begin{array}{|c|c|c|c|} \hline & & \dots & 6n \text{ elements} & \dots & & \\ \hline \end{array}$$

Multiple Particles

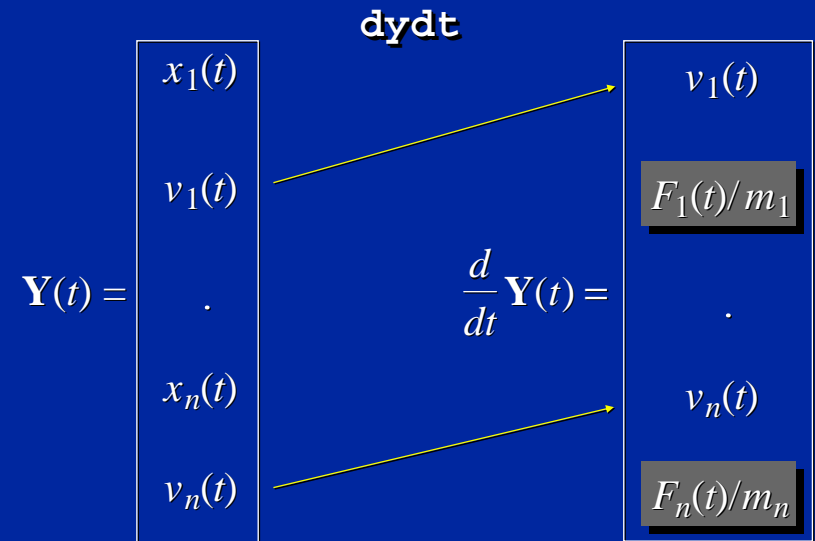


ODE solution

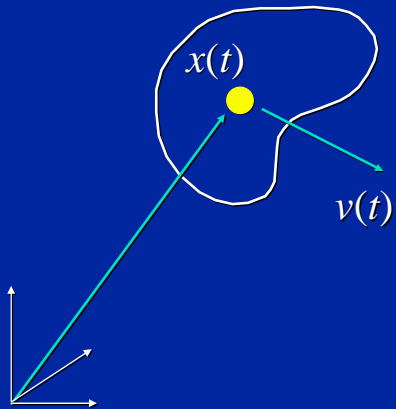




```
void dydt(double t, double y[],
          double ydot[])
```



Rigid Body State

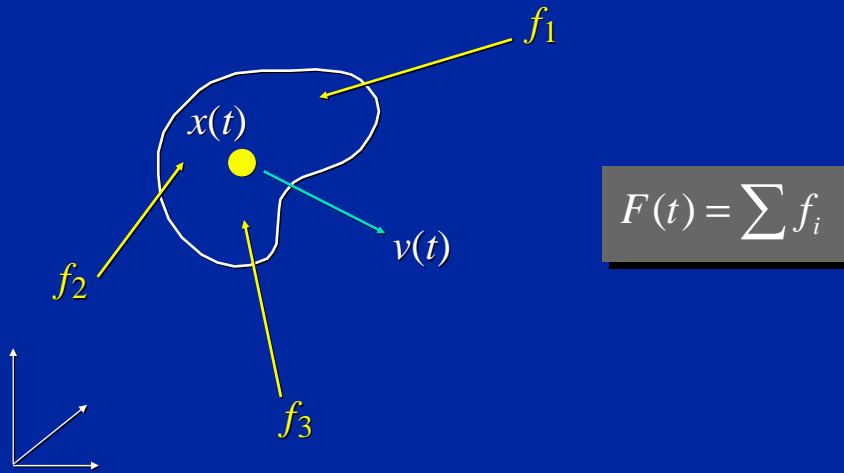


$$\mathbf{Y} = \begin{pmatrix} x(t) \\ ? \\ v(t) \\ ? \end{pmatrix}$$

Rigid Body Equation of Motion

$$\frac{d}{dt} \mathbf{Y} = \frac{d}{dt} \begin{pmatrix} x(t) \\ ? \\ Mv(t) \\ ? \end{pmatrix} = \begin{pmatrix} v(t) \\ ? \\ F(t) \\ ? \end{pmatrix}$$

Net Force



Orientation

We represent orientation as a rotation matrix[†] $R(t)$. Points are transformed from body-space to world-space as:

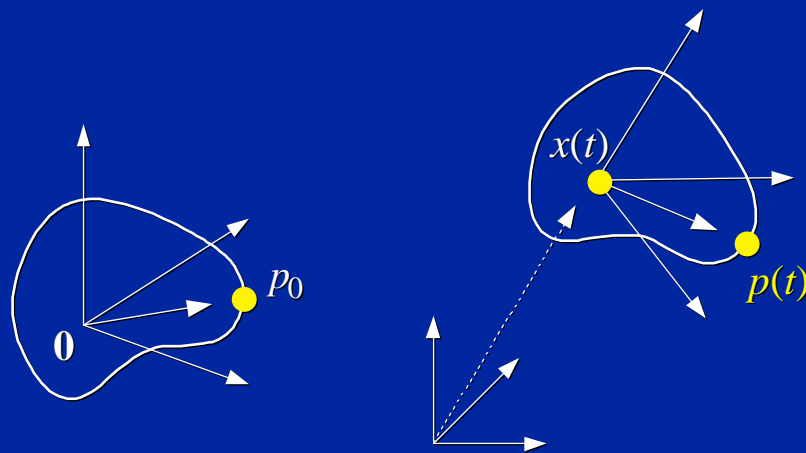
$$p(t) = R(t)p_0 + x(t)$$

[†]Actually, we use quaternions.

Angular Velocity

We represent angular velocity as a vector $\omega(t)$, which encodes both the axis of the spin and the speed of the spin.

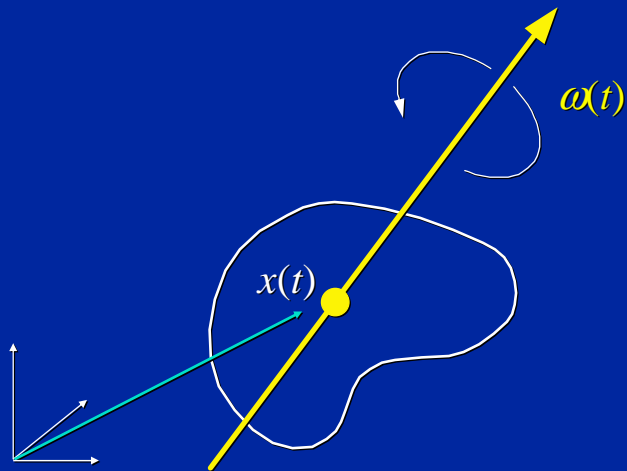
How are $R(t)$ and $\omega(t)$ related?



body space

world space

Angular Velocity Definition



Angular Velocity

$\dot{R}(t)$ and $\omega(t)$ are related by

$$\frac{d}{dt} R(t) = \begin{pmatrix} 0 & -\omega_z(t) & \omega_y(t) \\ \omega_z(t) & 0 & -\omega_x(t) \\ -\omega_y(t) & \omega_x(t) & 0 \end{pmatrix} R(t)$$

$(\omega(t))^*$ is a shorthand for the above matrix

Rigid Body Equation of Motion

$$\frac{d}{dt} \mathbf{Y} = \frac{d}{dt} \begin{pmatrix} x(t) \\ R(t) \\ Mv(t) \\ \langle \omega(t) \rangle \end{pmatrix} = \begin{pmatrix} v(t) \\ \omega(t)^* R(t) \\ F(t) \\ ? \end{pmatrix}$$

Need to relate $\dot{\omega}(t)$ and mass distribution to $F(t)$.

Inertia Tensor

$$I(t) = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$

diagonal terms

off-diagonal terms

$$I_{xx} = M \int_V (y^2 + z^2) dV$$

$$I_{xy} = -M \int_V xy dV$$

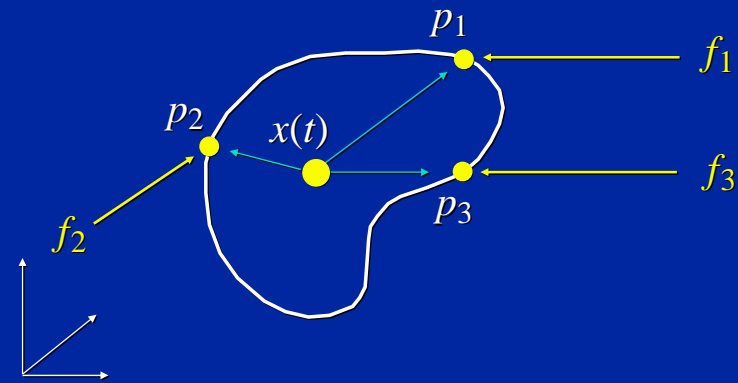
Rigid Body Equation of Motion

$$\frac{d}{dt} \mathbf{Y} = \frac{d}{dt} \begin{pmatrix} x(t) \\ R(t) \\ \boxed{Mv(t)} \\ \boxed{I(t)\omega(t)} \end{pmatrix} = \begin{pmatrix} v(t) \\ \omega(t)^* R(t) \\ F(t) \\ \tau(t) \end{pmatrix}$$

$P(t)$ – linear momentum

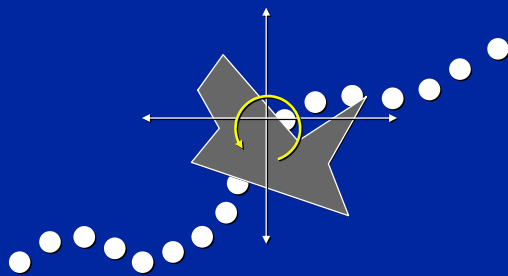
$L(t)$ – angular momentum

Net Torque



$$\tau(t) = \sum (p_i - x(t)) \times f_i$$

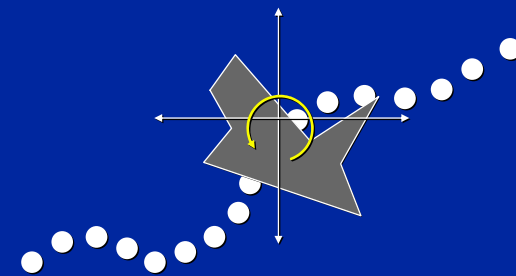
Inertia Tensors Vary in World Space...



$$I_{xx} = M \int_V (y^2 + z^2) dV$$

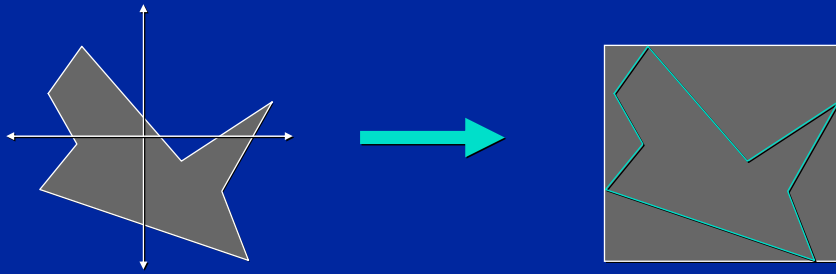
$$I_{xy} = -M \int_V xy dV$$

... but are Constant in Body Space



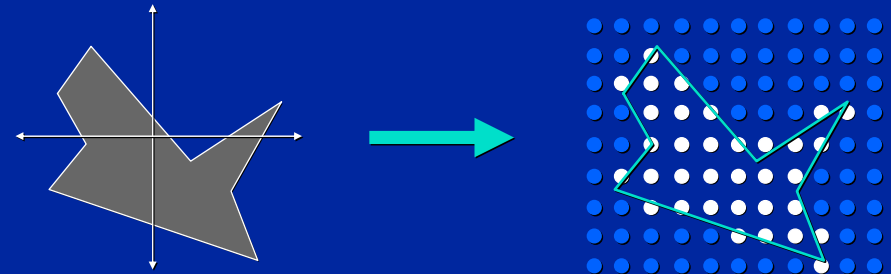
$$I(t) = R(t) I_{\text{body}} R(t)^T$$

Approximating I_{body} —Bounding Boxes



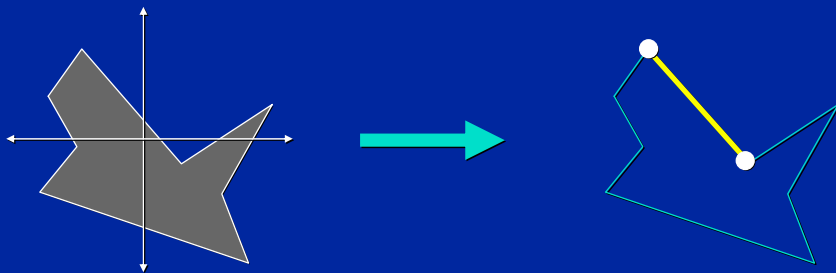
Pros: Simple.
Cons: Bounding box may not be a good fit.
Inaccurate.

Approximating I_{body} —Point Sampling



Pros: Simple, fairly accurate, no B-rep needed.
Cons: Expensive, requires volume test.

Computing I_{body} —Green's Theorem (Twice!)



Pros: Simple, exact, no volumes needed.
Cons: Requires B-rep.
Code: <http://www.acm.org/jgt/papers/Mirtich96>

Computing I_{body} at the center of mass

The inertia tensor should be relative to the body's center of mass, e.g:

$$I_{xx} = I'_{xx} - m(r_y^2 + r_z^2)$$

$$I_{xy} = I'_{xy} - mr_x r_y$$

Rigid Body Equation of Motion

$$\frac{d}{dt} \mathbf{Y} = \frac{d}{dt} \begin{pmatrix} x(t) \\ R(t) \\ \boxed{Mv(t)} \\ \boxed{I(t)\omega(t)} \end{pmatrix} = \begin{pmatrix} v(t) \\ \omega(t)^* R(t) \\ F(t) \\ \tau(t) \end{pmatrix}$$

$P(t)$ – linear momentum

$L(t)$ – angular momentum

Related Research

- Interactive control of rigid-body simulations