Projections

Reading

Foley et al. Chapter 6

Optional


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3D Geometry Pipeline

Image space (Window space)

Normalized device space (Screen space)

Image space (Screen space)

Normalized device space (Screen space)

Projected transformation, scale, translate

Project, scale, translate

Model space (Object space)

World space (Object space)

Eye space (View space)

scale, translate, rotate...

rotate, translate

rotate, translate
Projections

Projections transform points in \( n \)-space to \( m \)-space, where \( m < n \).

In 3D, we map points from 3-space to the projection plane (PP) along projectors emanating from the center of projection (COP).

There are two basic types of projections:
- Perspective - distance from COP to PP finite
- Parallel - distance from COP to PP infinite

Perspective vs. parallel projections

Perspective projections pros and cons:
+ Size varies inversely with distance - looks realistic
– Distance and angles are not (in general) preserved
– Parallel lines do not (in general) remain parallel

Parallel projection pros and cons:
– Less realistic looking
+ Good for exact measurements
+ Parallel lines remain parallel
– Angles not (in general) preserved

Parallel projections

For parallel projections, we specify a direction of projection (DOP) instead of a COP.

There are two types of parallel projections:
- Orthographic projection — DOP perpendicular to PP
- Oblique projection — DOP not perpendicular to PP

There are two especially useful kinds of oblique projections:
- Cavalier projection
  • DOP makes 45° angle with PP
  • Does not foreshorten lines perpendicular to PP
- Cabinet projection
  • DOP makes 63.4° angle with PP
  • Foreshortens lines perpendicular to PP by one-half

Orthographic Projections
Oblique projections

Two standard oblique projections:

- Cavalier projection
  - DOP makes 45 angle with PP
  - Does not foreshorten lines perpendicular to PP
- Cabinet projection
  - DOP makes 63.4 angle with PP
  - Foreshortens lines perpendicular to PP by one-half

Oblique Projections

Oblique projection geometry

Projection taxonomy

Projection taxonomy

Properties of projections

The perspective projection is an example of a projective transformation.

Here are some properties of projective transformations:

- Lines map to lines
- Parallel lines don’t necessarily remain parallel
- Ratios are not preserved
Coordinate systems for CG

- **Model space** — for describing the objections (aka “object space”, “world space”)
- **World space** — for assembling collections of objects (aka “object space”, “problem space”, “application space”)
- **Eye space** — a canonical space for viewing (aka “camera space”)
- **Screen space** — the result of perspective transformation (aka “normalized device coordinate space”, “normalized projection space”)
- **Image space** — a 2D space that uses device coordinates (aka “window space”, “screen space”, “normalized device coordinate space”, “raster space”)

A typical eye space

- **Eye**
  - Acts as the COP
  - Placed at the origin
  - Looks down the z-axis
- **Screen**
  - Lies in the PP
  - Perpendicular to z-axis
  - At distance d from the eye
  - Centered on z-axis, with radius s

Q: Which objects are visible?

**Eye space ➔ screen space**

Q: How do we perform the perspective projection from eye space into screen space?

Using similar triangles gives:

\[
\begin{align*}
X &= \frac{x}{d} \\
Y &= \frac{y}{d} \\
Z &= \frac{z}{d} \\
W &= \frac{1}{d}
\end{align*}
\]

We can write this transformation in matrix form:

\[
\begin{bmatrix}
X \\
Y \\
Z \\
W
\end{bmatrix} = MP =
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1/d & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z \\
1
\end{bmatrix}
\]

Perspective divide:

\[
\begin{bmatrix}
X/W \\
Y/W \\
Z/W \\
W/W
\end{bmatrix} = \frac{1}{d}
\begin{bmatrix}
x \\
y \\
z/d \\
d
\end{bmatrix}
\]

**Eye space ➔ screen space, cont.**
Projective Normalization

After perspective transformation and perspective divide, we apply parallel projection (drop the z) to get a 2D image.

Perspective depth

Q: What did our perspective projection do to $z$?

Often, it’s useful to have a $z$ around — e.g., for hidden surface calculations.

Vanishing points

Under perspective projections, any set of parallel lines that are not parallel to the PP will converge to a vanishing point.

Vanishing points of lines parallel to a principal axis $x$, $y$, or $z$ are called principal vanishing points.

Vanishing points

A line

$$P + tv = \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} + t \begin{bmatrix} v_x \\ v_y \\ v_z \\ 0 \end{bmatrix}$$

After perspective transformation we get:

$$t' = \frac{p_x + tv_x}{d}$$
$$t' = \frac{p_y + tv_y}{d}$$
$$t' = \frac{p_z + tv_z}{d}$$
Vanishing points, cont’d

Dividing by \( w \):

\[
\begin{bmatrix}
{t'} \\
{l'} \\
1
\end{bmatrix} = \begin{bmatrix}
{p_x + tv_x} \\
{p_y + tv_y} \\
{p_z - tv_z}
\end{bmatrix} \begin{bmatrix}
{p_x} \\
{p_y} \\
{p_z}
\end{bmatrix} \begin{bmatrix}
{d} \\
{d} \\
1
\end{bmatrix}
\]

Letting \( t \) go to infinity:

\[
\lim_{t \to \infty} \frac{p_x + tv_x}{p_y - tv_y} d = \lim_{t \to \infty} \frac{p_y + tv_y}{p_z - tv_z} d = \frac{v_x d}{v_y} = \frac{v_y d}{v_z}
\]

We get a point! This point does not depend on \( \hat{P} \) so any line in the direction \( v \) will go to the same point.

Types of perspective drawing

Perspective drawings are often classified by the number of principal vanishing points.

- One-point perspective — simplest to draw
- Two-point perspective — gives better impression of depth
- Three-point perspective — most difficult to draw

All three types are equally simple with computer graphics.

General perspective projection

In general, the matrix

\[
\begin{bmatrix}
1 \\
1 \\
q & r & s
\end{bmatrix}
\]

performs a perspective projection into the plane

\[px + qy + rz + s = 1\]

Q: Suppose we have a cube \( C \) whose edges are aligned with the principal axes. Which matrices give drawings of \( C \) with

- one-point perspective?
- two-point perspective?
- three-point perspective?

World Space Camera
Hither and yon planes

In order to preserve depth, we set up two planes:

- The **hither** (near) plane
- The **yon** (far) plane

![Diagram of hither and yon planes]

**Summary**

Here’s what you should take home from this lecture:

- The classification of different types of projections.
- The concepts of vanishing points and one-, two-, and three-point perspective.
- An appreciation for the various coordinate systems used in computer graphics.
- How the perspective transformation works.

![Projection taxonomy diagram]